OBESITY EXPERTISE

A first-principles curriculum in 196 books

BOOK 2 3rd of 196 in the series

Mathematics for modelling and inference · Rung 1

Reading the equations in a paper instead of skipping them

RATES, TOTALS, TABLES OF NUMBERS, CHANCE, AND WHERE A SYSTEM SETTLES

Dr. Harsh Maheshwari

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drharshmaheshwari.com · version 1.1

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This book is for learning. It is not medical, legal or financial advice, and it does not replace a qualified professional who knows your situation.

About the author



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Harsh Maheshwari completed his MBBS at King George's Medical University (KGMU), Lucknow, and is a resident in MD Community Medicine at the All India Institute of Medical Sciences (AIIMS), Raipur. His MD thesis, on chrononutrition and obesity, was selected by the Indian Council of Medical Research (ICMR) for funding in 2026.

He learns best from first principles: rebuilding an idea from what came before it until the formula feels like the only reasonable answer. This series grew out of that habit.

His long-term aims are to reduce obesity in India and to help make India healthy.

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Introduction

Why this book exists

Most papers on body weight carry a few equations, and most readers skip them. The argument is often in those lines. Book 0 named the derivative and the integral and asked you to work out neither. This book asks you to work them out, on body stores rather than on made-up curves. It adds the few tools that the equations in these papers keep using: tables of numbers multiplied together, the arithmetic of chance, and the point where a changing system stops changing.

How to read it

Read Book 0 and Book 1 first, or at least the sections this book names as it goes. Then read this one in order: each section assumes everything before it. Do the practice problems in each section before you read on, with a pen and a calculator. Keep a notebook open as you go. By the end it should hold ten equations from papers you have read, each written out in words. What this book does not teach yet: the product and chain rules, partial derivatives, likelihood, confidence intervals, hypothesis tests, regression and causal inference. Later books in the series take them up; "inference" in the title is where this subject's rungs are heading.

Every section has the same shape. **Definition** states the idea exactly. **In plain terms** says it again in everyday words. **Illustration** works through one real case, usually with a source you can open yourself. **Where this picture breaks** says where the illustration stops being true. **Must-know points** are what should change what you do. Then come **exercises** and, in most sections, a graded **practice set**.

Do the exercises before you look anything up. The worked answers are in the appendix at the back, under the same numbers. The references for each Part follow that Part. The glossary lists every term of art in the plain words it was first given. The last pages list all 196 books in the series and show where this one sits.

If you are new to this

On a first reading, read each section's definition, plain-terms explanation, illustration and must-know points, and do the exercises. Worked derivations can be skimmed and revisited once the idea feels familiar. Keep a calculator beside you: every worked answer can be followed with pen, paper and a calculator. Anything beyond the stated prerequisites is explained where it first appears, or marked as beyond this book. The **Symbols used in this book** page lists every sign and Greek letter with how to read it and where it is first used.

How to send feedback

If something is wrong, unclear, or could be better, please write to **contact@drharshmaheshwari.com**. Put the book number, the version and the section in the subject line (for example: *Book 2, v1.1, section C5*). Say what the text says, what you think it should say, and, if you can, the source.

Corrections are made in new versions. The version number and the date of the last update are on the copyright page of every book, and the newest version of every book is at **drharshmaheshwari.com**.

> **WHERE THIS BOOK SITS**
> Book 2 of the series' 196 books (Book 0 and 195 rung books). Before it: Book 1, Energy balance and dynamic weight-change modelling · Rung 1. After it: Book 3, Qualitative and mixed methods · Rung 1. The full list is at the back.

Symbols used in this book

Every sign and Greek letter this book prints, how to read it aloud, what it means in this series, and the section where it first appears. Each is also explained where it is first used.

Symbol	Read as	Meaning	First in
SIGNS			
×	times	Multiplication	1 · Reading the symbols in a methods section
Δ	delta, change in	A change or difference: Δt is a change in time	1 · Reading the symbols in a methods section
Σ	sigma, the sum of	Add up everything that follows: $\Sigma x = x_1 + x_2 + \ldots + x_n$	1 · Reading the symbols in a methods section
−	minus	Subtraction, or a negative number	1 · Reading the symbols in a methods section
′	prime	A modified version of a quantity, or its rate of change	2 · The derivative: the rate at an instant, computed
∫	the integral of	The area under a curve, or the total accumulated over a range	5 · The integral of a rate is the change in the stock
÷	divided by	Division	8 · Equilibrium: where the rate is zero
≥	greater than or equal to	An inequality that includes its boundary	13 · A random variable and its distribution
≤	less than or equal to	An inequality that includes its boundary	13 · A random variable and its distribution
±	plus or minus	A range either side of a value: 11.8 ± 0.6 means 11.2 to 12.4	15 · Variance, and why independent errors add as squares
§	section	A numbered section of a document, as in a statute's §5 or a book's §7.3	17 · Four kinds of equation, and the notebook entry
MARKS ON LETTERS			
x^2, 10^{-3}	superscript	Small raised characters are powers: $x^2 = x \times x$, and $10^{-3} = 0.001$	Before you start
GREEK LETTERS			
ρ	rho	A correlation in a population, or a density	1 · Reading the symbols in a methods section
δ	delta	A small change or step	1 · Reading the symbols in a methods section

Symbol	Read as	Meaning	First in
ε	epsilon	A small quantity; in a model, the error term	1 · Reading the symbols in a methods section
β	beta	In statistics a regression coefficient, or the Type II error rate; in biology beta cells, β-oxidation	1 · Reading the symbols in a methods section
τ	tau	A time constant, or Kendall's rank correlation	8 · Equilibrium: where the rate is zero
σ	sigma	A population standard deviation (σ^2 its variance)	15 · Variance, and why independent errors add as squares

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Before you start

This book uses the following Book 0 sections. Each is summarised where it is first needed; work through Book 0 itself if a summary is not enough.

- Book 0, A1 · Counting, place value, and what a calculator is actually doing
- Book 0, A4 · Percentages, and the percentage of a percentage
- Book 0, A5 · Ratios, rates and proportions
- Book 0, A6 · Powers, roots and scientific notation
- Book 0, A7 · Logarithms — what they are for before how they work
- Book 0, B1 · What a unit is, and the SI base and derived units
- Book 0, B3 · Dimensional analysis as an error check
- Book 0, B4 · Energy units: joule, kilojoule, calorie, kilocalorie
- Book 0, B6 · Body-size units: kg, m, cm and kg/m^2
- Book 0, C1 · Variables and algebra: what a letter stands for
- Book 0, C2 · Rearranging an equation
- Book 0, C3 · Two equations at once
- Book 0, C4 · Functions: input, rule, output
- Book 0, C5 · Graphs: axes, scale, slope and intercept
- Book 0, C6 · Recognising shapes: linear, exponential, saturating
- Book 0, C7 · Rate of change, average and instantaneous, without the machinery
- Book 0, C8 · Accumulation, a running total, without the machinery
- Book 0, C9 · Stocks and flows
- Book 0, D1 · What a probability is
- Book 0, D2 · Counting outcomes; independence
- Book 0, D4 · Variation: what differs and by how much
- Book 0, D5 · Average and spread, and why the average is not the person
- Book 0, D6 · Sampling: how a part can tell you about a whole
- Book 0, D7 · Random error and systematic error as different things
- Book 0, E2 · Chemical energy, combustion, and the bomb calorimeter
- Book 0, E4 · Heat and temperature; how heat moves
- Book 0, E8 · Organs and systems that matter for metabolism

- Book 0, F1 · Reading a table
- Book 0, F3 · Checking a reference
- Book 0, F4 · Argument: premise, inference, conclusion

Rung 1 — Introductory

1 · Reading the symbols in a methods section

DEFINITION. A symbol in an equation stands for a quantity, and the paper fixes which quantity in words: in a clause beginning “where” after the equation, or in the sentences around it.

A subscript is a small letter or number set low after a symbol, and it is part of the symbol’s name. A subscript never multiplies.

Every term added to or subtracted from another, and each side of an equals sign, carries the same unit. [1, 2, 3, 3, 4, 5, 6, 7, 8]

IN PLAIN TERMS. Each symbol stands for a quantity, and the paper says which one, in words, somewhere near the equation. Often that sentence starts with the word “where”. Find it before you read the equation.

Two symbols side by side are multiplied: 2i is 2 times i. Papers also write times as × or *. They write “greater than” as >, so x > 0 says x is above zero.

f(x) is the output of a rule called f, not f times x (Book 0, C4). A letter before brackets can mean either. If the paper calls the letter a parameter, a fixed number, it multiplies. If the brackets hold the input, such as t, it is a rule’s output.

Subscripts. A subscript is a small letter or number set low after a symbol. It is part of the symbol’s name, and it does one of two jobs.

It can label. Hall and colleagues write E for energy and set a small I, O or S after it. E with I is energy intake (EI). E with O is energy output (EO). E with S is energy stored (ES).

For output this book writes energy expenditure (EE), the name Book 1 used. EE and Hall’s EO are the same quantity.

Or it can count. BW with a small i after it is the body weight in interval number i. BW with a small 0 is the weight at the start, the baseline.

This book writes a subscript on the line, straight after the symbol: EI, BWi, BW0. A letter or digit written after a symbol is a label. BW0 is the starting weight, not BW times 0.

So side by side means multiply, with exceptions. A name of several letters (EI, BW, UGE) is one name. Δ or d in front of a symbol is part of it. A subscript is a label. A rule name before brackets gives the rule's output. The "where" clause tells you which.

Greek letters. They are letters like x and y, with names. Say the name out loud as you read.

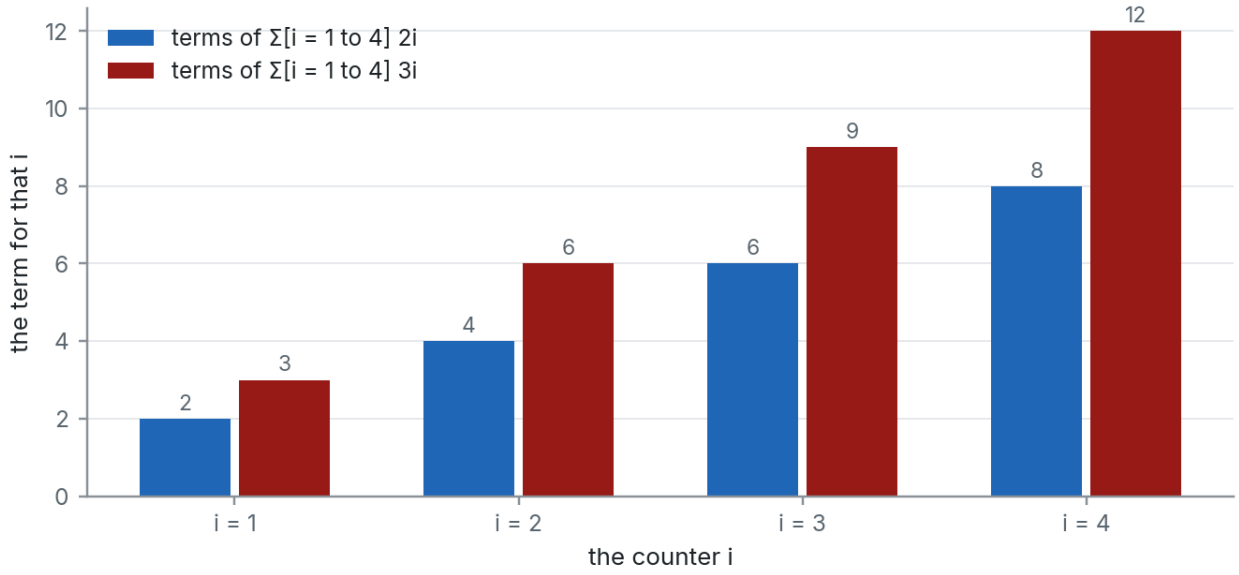
lette r	its name	say it	what one paper in this book uses it for
ρ	rho	"roe"	an energy density, energy per kilogram
Δ	capital delta	"delta"	the change in whatever follows it
δ	small delta	"delta"	a physical activity parameter: the energy cost of activity per kg of body weight (Polidori)
ε	epsilon	"ep-sih-lon"	how expenditure depends on weight (Polidori)
Σ	capital sigma	"sigma"	add up

Each paper decides what its letters mean, in its own "where" clause.

Sigma. Σ means "add up". It comes with a counter, usually i, and the counter's first and last values. On one line this book writes them in square brackets after it. Say the sum in words, then write it out.

```
Σ[i = 1 to 4] 2i
= the sum, for i from 1 to 4, of 2 times i
= 2 times 1 plus 2 times 2 plus 2 times 3 plus 2 times 4
= 2 plus 4 plus 6 plus 8
= 20
```

Units. Every term that is added or subtracted must carry the same unit, and so must the two sides of an equals sign (Book 0, B3). Use that to find the unit of a symbol the paper never states.



Two sigma sums drawn as their terms. The term for each i stands at i, so the counter only says where a term sits. The sum is the heights added up: 2 + 4 + 6 + 8 = 20 for Σ[i = 1 to 4] 2i, and 3 + 6 + 9 + 12 = 30 for Σ[i = 1 to 4] 3i.

ILLUSTRATION. Hall and colleagues (2012) wrote a consensus statement on energy balance — a text a panel of experts agreed on together. In it they write the energy balance equation like this.

```
ES = EI minus EO
```

ES is the energy stored in the body. The same paper puts the energy held in a lean adult's fat cells at about 130,000 kcal. So the stores are well over 130,000 kcal, and the equation seems to say intake minus output is too.

Run the unit check before you believe that. The paragraph says what the terms are measured in: "All of these terms are expressed as energy per unit of time." So EI and EO are in kilocalories a day.

The left side has to match. So ES here cannot be the stores. Their unit, kcal, has no time in it.

The paper says it in words: "ES is the rate of change in the body's macronutrient stores." The unit check caught the switch before the words did.

This book keeps the two apart. ES is the store, in kilocalories. Its rate of change is dES/dt, said "dee ES by dee t". In this book's letters Hall's equation says this. The rate of change of stored energy equals energy intake minus energy expenditure.

```
dES/dt = EI minus EE
```

Polidori and colleagues (2016) worked out how much people's eating changed during a trial of a drug. The drug raises urinary glucose excretion (UGE): the body loses glucose in the urine. Here is their Equation 1.

```
ΔEIi = ρ (dBWi/dt) + ε(BWi - BW0) + (Δδ/(1 - β)) BW0 + UGE
```

Do not read it left to right. Make a table with one row for each symbol. Fill the "words" column from the paper's own sentences. Leave a cell empty rather than guess.

The words are in the Methods. "We calculated the changes in energy intake, ΔEI, for each subject". One input was "the change of body weight versus baseline over each interval, (BWi − BW0)". The other was "the rate of change of body weight over each interval, dBWi/dt". That rate is an average over the whole interval, not a rate at one instant.

Then "The model parameter ρ was the effective energy density associated with the body weight change". Here a parameter is a number the paper keeps the same for every person and every interval. (Book 0, D6 used the word for a number describing a whole population; papers use both senses.) Then "ε defined how energy expenditure depends on body weight". And "Δδ represents changes in physical activity". And "The parameter β accounts for the adaptation of energy expenditure during a diet perturbation".

UGE is not spelled out beside the equation. When a short form is not spelled out, search the paper for words that fit its letters.

Start with the ones you know. The results give changes in intake in kilocalories a day, so ΔEIi is in kcal a day. Body weight is in kilograms. Time between weighings was counted in days, so dBWi/dt is in kilograms a day.

Every term on the right must also be in kcal a day. Work out each unknown unit from that.

```
ρ times (kg a day) = kcal a day, so ρ is in kcal per kg
ε times (kg) = kcal a day, so ε is in kcal a day per kg
1 minus β: a bare 1 can only lose a bare number, so β has no unit
Δδ times (kg) = kcal a day, so Δδ is in kcal a day per kg
```

That fits the paper's words: activity has "an energy cost that is proportional to body weight".

symbol	say it	the paper's words	unit
ΔEIi	delta E I sub i	change in energy intake, interval i	kcal a day
BWi	B W sub i	body weight, interval i	kg
BW0	B W sub zero	body weight at baseline	kg
dBWi/dt	dee B W sub i by dee t	rate of change of body weight, interval i	kg a day
ρ	rho	effective energy density of the weight change	kcal per kg (worked out)
ε	epsilon	how expenditure depends on weight	kcal a day per kg (worked out)
Δδ	delta delta	change in physical activity	kcal a day per kg (worked out)
β	beta	adaptation of expenditure to a diet change	none (worked out)
UGE	U G E	energy lost in urinary glucose excretion	kcal a day

Elsewhere the paper says urinary glucose excretion "was assumed to be ~90 g/day", about 90 grams a day. Grams a day cannot be added to kcal a day. The paper says what the term holds: "UGE represents the energy losses". Its Table 1 gives that loss as 360 kcal a day. It gives no energy per gram, so do not put 90 into this equation.

Now go the other way, from a written-out sum back to sigma. Take 3 plus 6 plus 9 plus 12. Each term is 3 times its place in the list, so it is Σ[i = 1 to 4] 3i. Check it by writing it out again.

```
3 times 1 plus 3 times 2 plus 3 times 3 plus 3 times 4 = 30
3 plus 6 plus 9 plus 12 = 30
```

Figures quoted in this illustration are from [9, 10].

WHERE THIS PICTURE BREAKS. The unit check shows that ES must be a rate in Hall's equation.

The table tells you what each symbol denotes and what unit it must carry. It does not tell you that the model is a good description of anyone.

MUST KNOW POINTS FOR YOU.

- Find the "where" clause before you read an equation. Then write a table: each symbol, its words from the paper, and its unit.
- One paper can use the same letters for two things. When the words are unclear, let the unit decide.
- A subscript is part of a name and never multiplies. Read it as "B W sub zero", out loud, until the habit holds.
- Work out a unit the paper does not print from the terms beside it. Whatever is subtracted from a bare 1 has no unit.
- A figure quoted in the text in one unit may enter the equation in another. Find the conversion before you put a number in.
- Matching units show that an equation is not malformed. They never show it is right, and they cannot tell apart two quantities with the same unit.
- Teach a trainee to read Σ aloud as "the sum, for i from 1 to n, of". Then have them write out the first two terms and the last.

EXERCISE 1 (interpretation). Polidori and colleagues (2016) model the change in eating after weight loss with this equation, their Equation 4.

```
ΔEI(t) = - kP × ΔBW(t)
```

The sentence after it reads: "where the parameter kP>0 quantifies the feedback strength". Their abstract reports eating "above baseline by ~100 kcal/day per kg of lost weight".

Write the symbol table for Equation 4: each symbol, how you say it, what it denotes, and its unit. Then say in one sentence what the minus sign does, and what number kP took.

PRACTICE. Eleven problems on this technique, easiest first. Work them on paper before turning to the answers, which are in the appendix at the back under this section's name.

1. Write this sum out term by term, then work it out.

```
Σ[i = 1 to 4] 3i
```

2. Write this sum in sigma notation.

```
2 plus 4 plus 6 plus 8 plus 10
```

3. Four values are x1 = 4, x2 = 7, x3 = 1 and x4 = 8.

Work out Σ[i = 1 to 4] xi, then Σ[i = 2 to 3] xi, then Σ[i = 1 to 4] 5xi.

4. An equation reads A = B + C × D. B is in kilocalories a day. D is in kilograms.

Work out the unit of A and the unit of C.

5. Three United Nations bodies wrote a report on human energy requirements (2004). They are the Food and Agriculture Organization (FAO), the World Health Organization (WHO) and the United Nations University (UNU). The report divides total energy expenditure (TEE) by basal metabolic rate (BMR), both in megajoules a day. The result is the physical activity level (PAL).

```
PAL = TEE/BMR
```

The same passage gives a man with a PAL of 1.75 and a mean BMR of 7.10 MJ/day.

Work out his TEE. Then give the unit of PAL, and say why it has that unit.

6. Hall (2008) writes: "The energy density of the body fat mass change, ρF = 39.5 MJ/kg". He uses ΔF for the change in body fat.

Say both symbols in words. Then take ΔF = -3 kg, a made-up figure. Work out ρF times ΔF, with its unit, and say what the sign means.

7. Polidori and colleagues (2016) define the length of each interval. They write "The interval length was t = (N−1)*T". Then they say what the letters are. "N = 2 was the number of body weight measurements per interval". And "T = 52 was the number of days between measurements".

Say what t, N and T each stand for, with their units. Work out t. Then work out what t would be with three measurements per interval, the same distance apart.

8. Here is a worked answer. Find the step that broke.

```
Equation 1 of Polidori and colleagues ends in "+ UGE"
UGE was assumed to be about 90 g/day
so the last term adds 90 to the change in energy intake
the other terms come to 250 kcal a day for this interval
250 plus 90 = 340, so ΔEI is 340 kcal a day
```

The 250 is made up for this problem.

9. Here is a worked answer. Find the step that broke.

```
one term of an equation is ε(BWi - BW0)
two symbols side by side are multiplied
so BW0 is BW times 0, which is 0
so the term is ε times BWi
with ε = 20 and BWi = 80, the term is 1600
```

Both numbers are made up for this problem.

10. A colleague reads Hall and colleagues' (2012) equation and says this.

> *ES = EI − EO. The paper also says a lean adult's fat cells hold about 130,000 kcal. So a lean adult has eaten 130,000 kcal more than they have spent.*

Decide what to check, check it, and say what your answer does not establish.

11. A trainee's notes on two papers say this.

> *Hall (2008) gives ρ = 39.5 MJ/kg. Polidori and colleagues (2016) have a ρ in Equation 1. So in their model each kilogram of weight change carries 39.5 MJ. That is about 9,441 kcal.*

Decide what to check, check it, and say what your answer does not establish.

2 · The derivative: the rate at an instant, computed

DEFINITION. The average rate of change of f over the interval from a to a + h is f(a + h) minus f(a), divided by h, for h not zero.

The derivative of f at a is the single number these averages settle on as h is taken closer and closer to zero, from above and from below. Where the two sides do not settle on one number, f has no derivative at a.

The derivative at a is written f′(a), said “f prime of a”.

The derivative carries the unit of f divided by the unit of the input.

The derivative belongs to a smooth curve, one with no jumps or corners, so that the averages settle. Repeated measurements are separate points with error in them. [11, 12, 13, 14]

IN PLAIN TERMS. Book 0 showed you the rate at an instant as the slope of a tangent, laid on a curve with a ruler. It named that rate the derivative (Book 0, C7). This section works the number out with arithmetic, no ruler needed.

A shorter interval gives a closer answer. So take the average rate over shorter and shorter intervals, all starting at the moment you care about. They settle on one number. That number is the rate at that instant.

Here is the procedure for a function f at an input a.

1. Pick a small step, h. Work out f(a + h) minus f(a), and divide by h.
2. Make h closer to zero: 1, then 0.1, then 0.01, then 0.001.
3. Do the same with h below zero: minus 1, minus 0.1, and so on. Those intervals end at a instead of starting there. (A negative divided by a negative is positive, Book 0, A1.)
4. If both lists close in on the same number, that number is very likely the derivative at a. The algebra, as below, settles it.

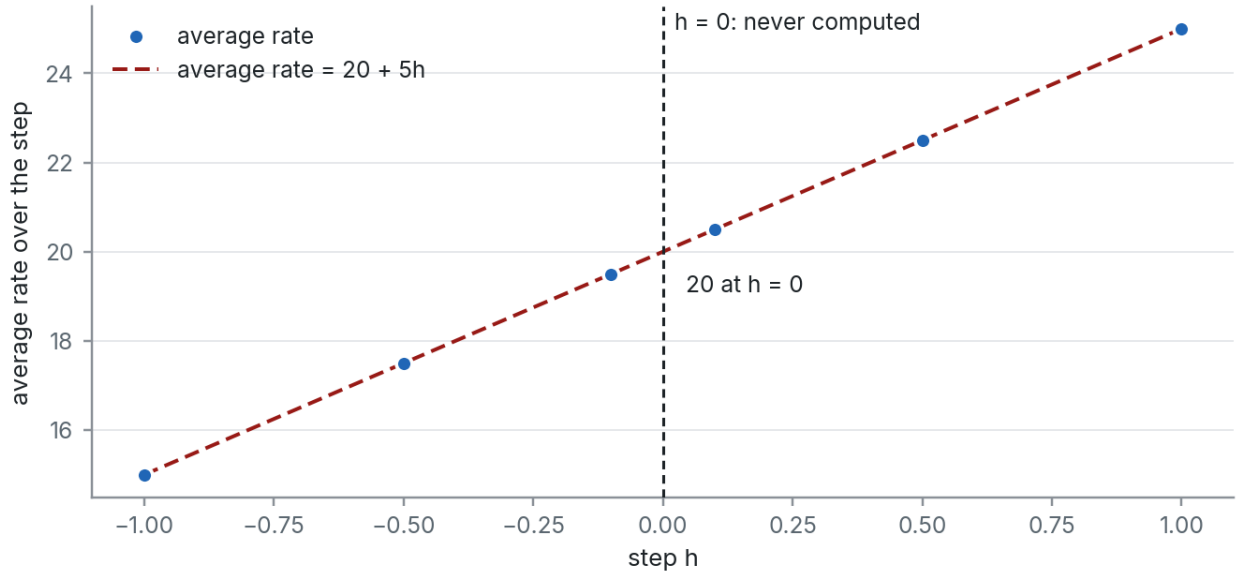
You never set h to zero. The number they close in on is called the limit.

The derivative has three names you will meet. f′(a), said “f prime of a”. dS/dt, said “dee S by dee t”. And in words, the rate of change of S with respect to t.

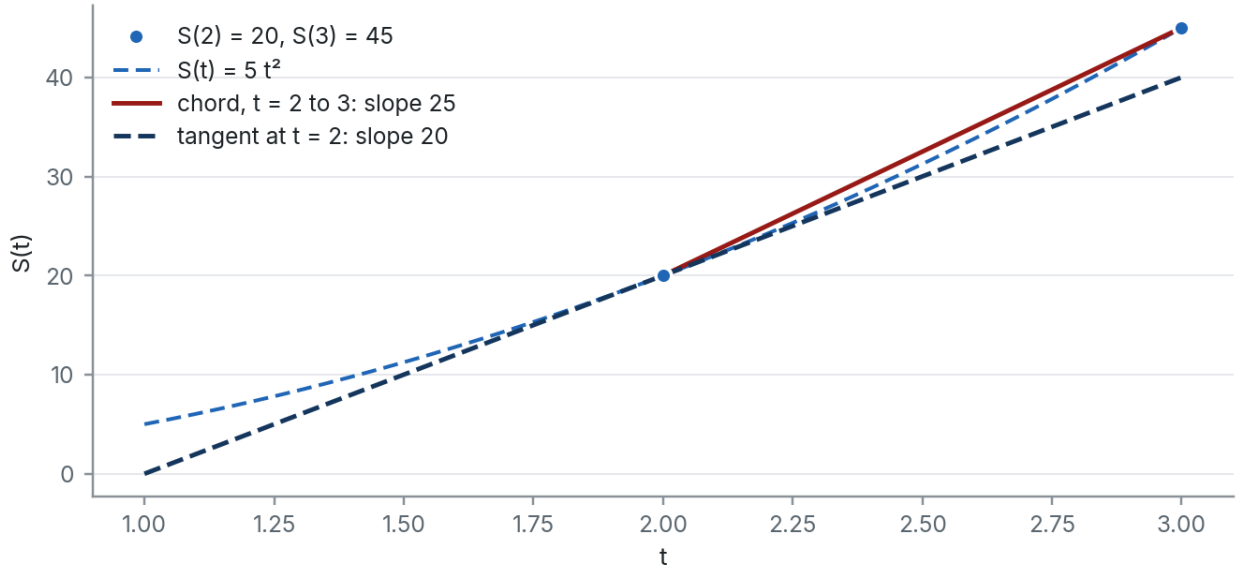
A weight in kilograms against time in days gives kilograms per day.

Its sign tells you the direction. Positive means rising at that instant. Negative means falling. Zero means neither, at that instant.

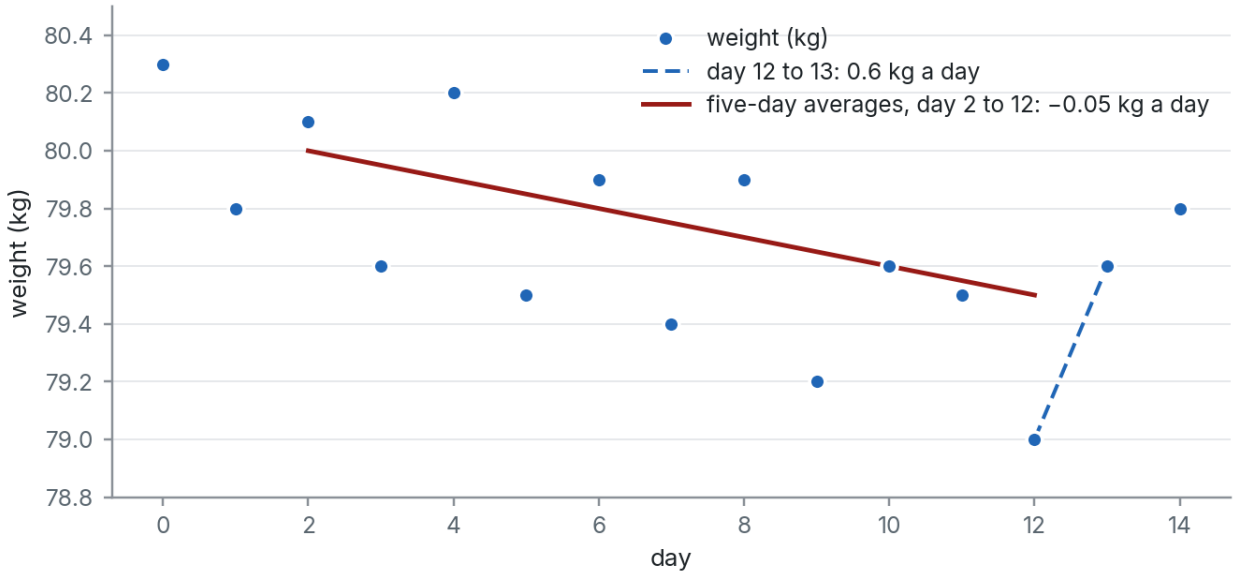
Read Hall and colleagues’ equation from the last section again in these terms. dES/dt is the rate of change of stored energy, in kilocalories a day. It is not the stored energy itself.



The average rate of $S(t) = 5\,t^2$ from t = 2 over a step h, for six steps from -1 to 1. Every average lies on the line 20 + 5h. As the step shrinks towards zero from either side, the averages close in on 20, the derivative at t = 2. The dashed line marks h = 0, which is never computed. The closer steps, down to 0.001, are in the table.



$S(t) = 5\,t^2$ near t = 2. The note's 25 is the slope of the chord from t = 2 to t = 3, an average over a whole unit of time. The tangent at t = 2 has slope 20, the rate at that instant, and the curve bends away above it.



Fifteen made-up morning weighings. The one-day rate from day 12 to day 13 is 0.6 kg a day upwards, and day to day the weight jumps by as much as 0.7 kg. The line from the average of days 0 to 4, 80.0 kg at day 2, to the average of days 10 to 14, 79.5 kg at day 12, falls at 0.05 kg a day. The side axis does not start at zero, so the swings look larger than they are.

ILLUSTRATION. A quantity follows the rule $S(t) = 5\,t^2$, which is 5 times t times t. A note says this.

> *S went from 20 at t = 2 to 45 at t = 3. So at t = 2 it is rising at 25 a unit of time.*

The 25 is the average over a whole unit of time, from t = 2 to t = 3. Test that by shrinking the interval.

Each row takes a step h from t = 2. Work out S(2 + h), subtract S(2), and divide by h.

step h	S(2 + h)	S(2 + h) minus S(2)	average rate
1	45	25	25
0.1	22.05	2.05	20.5
0.01	20.2005	0.2005	20.05
0.001	20.020005	0.020005	20.005
-0.001	19.980005	-0.019995	19.995
-0.01	19.8005	-0.1995	19.95
-0.1	18.05	-1.95	19.5
-1	5	-15	15

Read down the last column. Both lists close in on 20. So the rate at t = 2 is 20 a unit of time, not 25.

Now see why it is exactly 20, with algebra. Each term of the first bracket multiplies the whole second bracket. That is the rule from Book 0, C1, used twice. The first working line shows it.

```
(2 + h) times (2 + h) = 2 times (2 + h) + h times (2 + h)
= 4 + 2h + 2h + h^2
= 4 + 4h + h^2
S(2 + h) = 5 times (4 + 4h + h^2) = 20 + 20h + 5 h^2
```

Subtract S(2), which is 20. Then divide by h.

```
S(2 + h) minus S(2) = 20h + 5 h^2
(20h + 5 h^2) divided by h = 20 + 5h
```

The average rate over a step h is 20 + 5h, for any h. As h shrinks, 5h shrinks with it, and what is left is 20.

Now the same move on real measurements, where it breaks. A made-up person weighs themselves each morning for fifteen days.

day	weight (kg)
0	80.3
1	79.8
2	80.1
3	79.6
4	80.2
5	79.5
6	79.9
7	79.4
8	79.9
9	79.2
10	79.6
11	79.5
12	79.0
13	79.6
14	79.8

Shrink the interval to one day, as you did for S. Take the last two days, then the pair one day earlier.

```
79.8 minus 79.6 = 0.2, over 1 day: 0.2 kg a day
79.6 minus 79.0 = 0.6, over 1 day: 0.6 kg a day
```

Rising at 0.6 kg a day. Test that against the body. Hall (2008) gives body fat 39.5 megajoules (MJ) a kilogram. A joint report of the Food and Agriculture Organization, the World Health Organization and the United Nations University (FAO/WHO/UNU) has a worked example. It puts the daily expenditure of an average lightly active woman of 30 to 50 at 8.26 MJ.

```
0.6 times 39.5 = 23.7
23.7 divided by 8.26 = 2.87
```

Adding 0.6 kg of fat in a day would take 23.7 MJ more than was spent. That is nearly three whole days of that expenditure. The one-day rate is measuring swings, not stores.

So with weighings, widen the interval instead. Average the first five days and the last five. Each average is placed at its middle day, the centre of the five days it covers. An average of neighbouring weighings, worked out again at each step along the series, is called a moving average.

```
80.3 plus 79.8 plus 80.1 plus 79.6 plus 80.2 = 400.0
400.0 divided by 5 = 80.0, the average weight at day 2
79.6 plus 79.5 plus 79.0 plus 79.6 plus 79.8 = 397.5
397.5 divided by 5 = 79.5, the average weight at day 12
79.5 minus 80.0 = -0.5
-0.5 divided by 10 = -0.05
```

Falling at about 0.05 kg a day over those ten days. Even this is still an average over an interval.

This section has now used three kinds of rate.

name	what it is	where it appears here
derivative	the number the averages close in on as h shrinks towards zero	$S(t) = 5\,t^2$ at $t = 2$: 20
finite difference	the change over a set interval, divided by its length	the note's 25; the one-day 0.6 kg a day
slope from noisy measurements	the change between averages of several readings, divided by the time between them	the five-day averages: -0.05 kg a day; Polidori's moving averages, in the exercise

Figures quoted in this illustration are from [15, 16, 10].

WHERE THIS PICTURE BREAKS. Below the interval over which water swings, a shorter interval gives a worse rate, not a better one. No source in this book gives that interval. Look for it in the Methods of the paper in front of you.

The fifteen weights are made up.

MUST KNOW POINTS FOR YOU.

- Before you call a rate "the rate now", ask how long the interval was.
- A rate of weight change taken from two weighings a day apart can be mostly water, not stored energy. This book's rule of thumb: take a rate from averages over weeks, not from neighbouring readings.
- A shorter interval is better only for a smooth curve.
- Never set h to zero to get the derivative. Let h shrink and watch where the answers go.
- Read dW/dt as a rate with a unit, kilograms per day, and read its sign first. It says nothing about how much has been lost so far.

- When a trial reports a rate of weight change, look in the Methods for the interval and the averaging used.

EXERCISE 1 (interpretation). Polidori and colleagues (2016) describe how they got one of their inputs. Here is their sentence.

> *the moving average of the measured body weight time course was used to calculate the rate of change of body weight over each interval*

They call that rate dBWi/dt.

Their weighings were 52 days apart.

Say what dBWi/dt denotes and what unit it carries. Then say why a rate taken over 52 days is safer than one taken from two weighings a day apart would be, and what it gives up.

PRACTICE. Twelve problems on this technique, easiest first. Work them on paper before turning to the answers, which are in the appendix at the back under this section's name.

1. A function has f(4) = 30 and f(4.5) = 33.5.

Work out the average rate of change of f from 4 to 4.5.

2. Take $f(t) = t^2$, which is t times t, and the input t = 3.

Work out the average rate from t = 3 to t = 3 + h for h = 1, 0.1 and 0.01. Do the same for h = -0.1 and h = -0.01. Then give the derivative at t = 3.

3. Say each of these aloud in words, and give its unit.

```
dW/dt, where W is body weight in kg and t is time in days
f'(10) = -3, where f is in kcal a day and its input is body weight in kg
```

4. Take $f(t) = 3\,t^2$ and the input t = 1.

Write out f(1 + h) minus f(1), divided by h, using algebra. Then find the derivative at t = 1.

5. Here are four values of a derivative. For each, say whether the function is rising, falling or neither at that input.

```
f'(0) = 4
f'(2) = 0
f'(5) = -0.3
f'(7) = -12
```

6. In the trial reported by Polidori and colleagues (2016), weighings were 52 days apart. Take a made-up participant who weighs 94.6 kg at one weighing and 92.0 kg at the next.

Work out the average rate of weight change over the interval, with its unit. Then give it in kilograms a week.

7. The FAO/WHO/UNU report (2004) puts the daily expenditure of an average lightly active woman of 30 to 50 at 8.26 MJ. Take a made-up woman with that expenditure, on a day when her metabolizable intake is 7.80 MJ.

Work out dES/dt, the rate of change of her energy stores, in MJ a day and then in kcal a day. Use 1 kcal = 4.184 kJ. Say what the sign means.

8. A patient weighs 92.0 kg today. Their weight is falling at 0.05 kg a day. Both figures are made up.

Estimate their weight 10 days from now, using the rate. Then do the same for 300 days from now, and say which of the two estimates you would trust less, and why.

9. Here is a worked answer. Find the step that broke.

> f(t) = t^2, find the derivative at t = 3
> f(3 + h) = (3 + h) times (3 + h) = 9 + h^2
> f(3 + h) minus f(3) = h^2
> h^2 divided by h = h
> as h shrinks, h shrinks to 0
> so the derivative at t = 3 is 0, and f is not changing there

10. Here is a worked answer. Find the step that broke.

> a patient weighed 79.0 kg on Monday morning and 79.6 kg on Tuesday morning
> the rate of weight change is 79.6 minus 79.0, over 1 day, = 0.6 kg a day
> a shorter interval gives a closer rate, so this is close to the rate at an instant
> so the patient is gaining 0.6 kg of body fat a day

11. A newspaper report says this about a new diet.

> *In the first week, people on the diet lost 2 kg. Kept up, that is over 100 kg in a year.*

The report and its 2 kg are made up for this problem. Decide what to compute, compute it, and say what your answer does not establish.

12. A clinic note reads as follows.

> *Weighed 76.2 kg yesterday and 76.2 kg today. dW/dt = 0. Her weight has stopped changing, so her intake now matches her expenditure.*

The weights are made up for this problem. Decide what to compute, compute it, and say what your answer does not establish.

3 · The few rules that do the differentiating

DEFINITION. Four rules give the derivative of any sum of constant multiples of whole-number powers of the input, without the shrinking-interval calculation.

The constant rule: the derivative of a constant is zero.

The power rule: for a positive whole number n, the derivative of x^n is n times $x^{(n \text{ minus } 1)}$.

The constant multiple rule: the derivative of k times a function is k times the derivative of the function.

The sum and difference rules: the derivative of a sum or difference of two functions is the sum or difference of their derivatives.

The derivative of a straight line is its slope.

The derivative of a product of two functions is not the product of their derivatives. The power rule does not apply to a constant raised to a power that varies, such as 2^t. [17, 18, 18, 19, 20, 21, 22, 23, 24]

IN PLAIN TERMS. The last section found a derivative by shrinking the interval. Four rules do the same job in a line, for any rule made of whole-number powers of the input, each times a number, added up.

First, a symbol. d/dx, said “dee by dee x”, means “the derivative with respect to x of what follows”. The input can have any letter: d/dt for time, d/dW for weight.

The constant rule. A constant has derivative zero.

$$\frac{d}{dx}(7) = 0$$

The power rule. For x raised to a whole number n, bring the n down in front and take one off the power.

$$\frac{d}{dx}(x^n) = n \text{ times } x^{(n \text{ minus } 1)}$$
$$\frac{d}{dx}(x^3) = 3\,x^2$$
$$\frac{d}{dx}(x^2) = 2x$$
$$\frac{d}{dx}(x) = 1 \text{ times } x^0 = 1$$

So x on its own has derivative 1.

The constant multiple rule. A number multiplying a function stays where it is. Take the derivative of the function, then multiply.

$$\frac{d}{dx}(5\,x^2) = 5 \text{ times } 2x = 10x$$

The sum and difference rules. Take each term’s derivative on its own, then add or subtract.

```
d/dx (x^3 + 5 x^2 minus 4) = 3 x^2 + 10x minus 0 = 3 x^2 + 10x
```

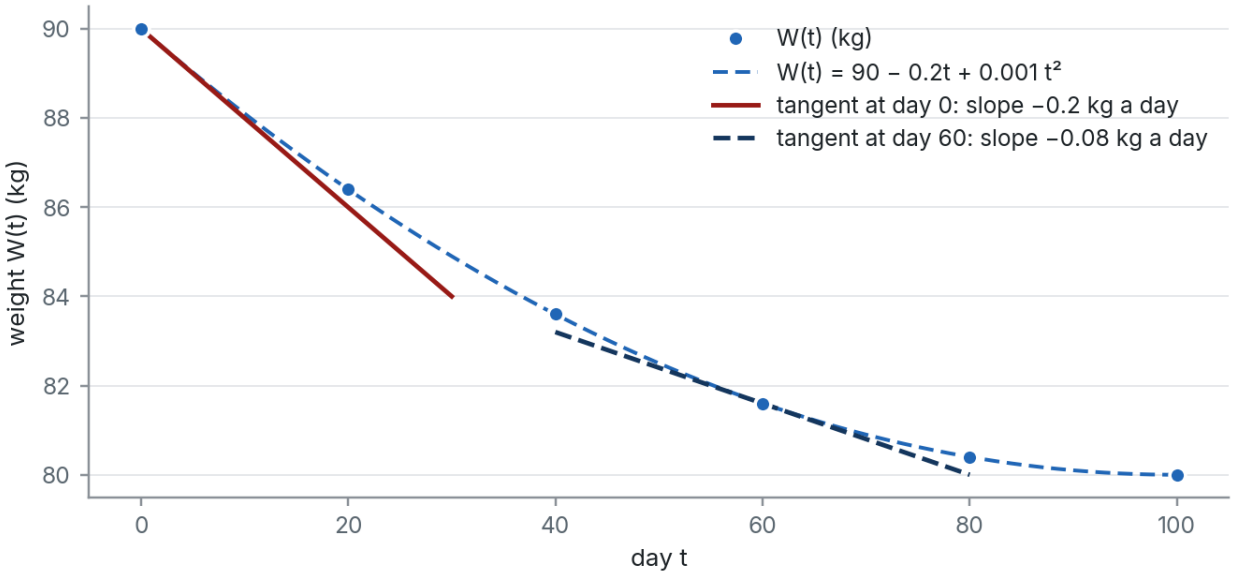
Put the rules together on a straight line, y = a + bx. The constant a goes to zero. So the derivative is b, the slope, at every x.

The second derivative. A derivative is itself a function, so you can take its derivative too. The result is the second derivative, the rate of change of the rate. It is written f″(x), said “f double prime of x”, or d^2y/dx^2, said “dee two y by dee x squared”.

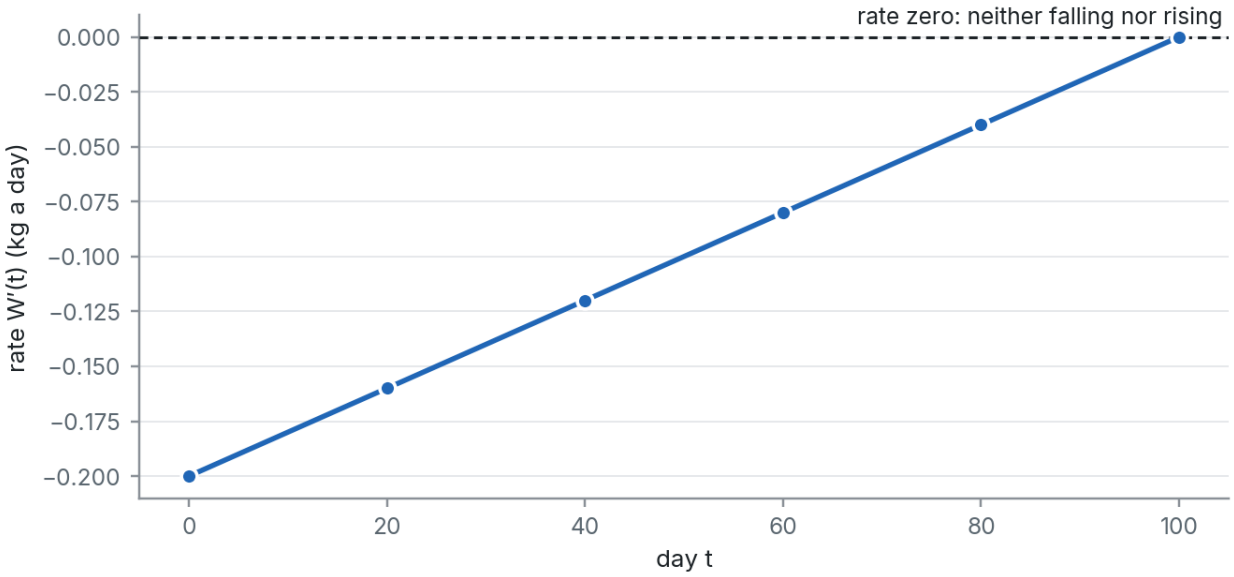
Its sign says whether the rate is rising or falling. When a quantity is falling, its rate is below zero. A positive second derivative then means the rate is rising towards zero. The quantity is still falling, only more slowly.

For values at equal steps, take the differences (Book 0, C6), then the differences of those. Their sign plays the part of the second derivative’s.

Two rules are not in this book. The derivative of a product is not the product of the derivatives: it needs the product rule. A rule applied to the output of another rule needs the chain rule. Both come in the rung-2 book of this subject. Until then, multiply a product out before you differentiate.



The made-up curve W(t) = 90 minus 0.2t + 0.001 t^2 over 100 days. Weight falls from 90 kg to 80 kg, and the fall slows as it goes: the tangent at day 0 has slope -0.2 kg a day and the tangent at day 60 only -0.08. The curve flattens because its second derivative, 0.002, is positive.



The same curve's first derivative, W′(t) = -0.2 + 0.002t. It is below zero on every day before day 100, so weight is falling throughout. It climbs towards zero at 0.002 kg a day per day, the second derivative, so the fall is slowing.

ILLUSTRATION. Take S(t) = 5 t^2 from the last section, where shrinking the interval gave 20 at t = 2.

```
S′(t) = 5 times 2t = 10t
S′(2) = 10 times 2 = 20
S′(3) = 10 times 3 = 30
```

The same 20, in two lines.

Hall and Guo (2017) describe expenditure rising with body weight "with a slope of about 20–30 kcal/d per kg". Take the low end and write it as a rule. W is body weight in kilograms, and EE is energy expenditure in kilocalories a day.

```
EE(W) = a + 20W
```

Here a is where the line meets the side axis. The quoted sentence does not give it. Differentiate term by term with respect to W.

```
d/dW (a) = 0, by the constant rule
d/dW (20W) = 20 times 1 = 20, by the constant multiple and power rules
dEE/dW = 0 + 20 = 20
```

Expenditure changes by 20 kcal a day for each kilogram of weight, at the low end of their range. You found the rate without knowing a, because a constant contributes nothing to a rate.

Kilocalories a day divided by kilograms is kcal a day per kg. This rate is not against time.

Hall and colleagues (2012) write that "weight change will slow over time due to passive compensatory changes in energy expenditure". Passive means it happens with no change in what the person does. A smaller body spends less, so the same intake leaves a smaller gap.

Here is a made-up curve with that shape, for 100 days of weight loss.

```
W(t) = 90 minus 0.2t + 0.001 t²
```

W is in kilograms and t is in days. A note on this curve, for a made-up patient, says this.

> *The second derivative is positive, so her weight is going back up.*

Differentiate once, term by term.

```
W′(t) = 0 minus 0.2 + 0.001 times 2t
W′(t) = -0.2 + 0.002t
```

Differentiate again.

```
W″(t) = 0 + 0.002 = 0.002
```

day t	W(t) (kg)	W′(t) (kg a day)
0	90	-0.2
20	86.4	-0.16
40	83.6	-0.12
60	81.6	-0.08
80	80.4	-0.04
100	80	0

The first derivative is below zero on every row until day 100. Weight is falling the whole time. The second derivative is positive, 0.002 kg a day per day. That says the rate is rising: from minus 0.2 towards zero. The fall is slowing.

Figures quoted in this illustration are from [25].

WHERE THIS PICTURE BREAKS. The straight line EE = a + 20W holds only over the range of weights the slope was measured on.

The curve W(t) is made up to fall and slow for 100 days, and holds nowhere else. Run it past day 100 and it turns.

MUST KNOW POINTS FOR YOU.

- When a paper gives expenditure as a straight line in weight, the change for a given weight change is the slope times that change. You need no intercept to work it out.
- A positive second derivative does not mean the quantity is rising. When weight is falling, a positive second derivative means the fall is slowing.
- Slowing weight loss is what the body's own responses predict: expenditure falls and appetite rises, even with no change in the treatment.

- The derivative of a product is not the product of the derivatives.
- These rules differentiate a formula exactly. They say nothing about whether the formula fits the body.
- Check a derivative you worked out by rule with the shrinking-interval method at one point.

EXERCISE 1 (teaching). A journalist has three minutes. A press release says weight loss on a drug "slowed after six months, showing its effect wears off". The press release is made up for this exercise.

Explain, without any symbols, what slowing weight loss does and does not tell you. Give them one sentence they could quote.

PRACTICE. Sixteen problems on this technique, easiest first. Work them on paper before turning to the answers, which are in the appendix at the back under this section's name.

1. Find each derivative.

```
d/dx (9)
d/dx (x^6)
d/dx (x)
```

2. Find each derivative.

```
d/dx (4 x^3)
d/dt (-2 t^5)
```

3. Find the derivative of f(x) = 3 x^4 minus 2x + 6.

4. Take f(t) = 2 t^3 minus t. Find f′(t), then its value at t = 2.

5. Take f(t) = t^3 minus 6 t^2 + 4.

Find f′(t) and f″(t). Then work out both at t = 1 and at t = 3, and say in words what f is doing at each.

6. Take the straight line y = 7 minus 0.4x. Find its first and second derivatives.

7. Hall and Guo (2017) give expenditure rising with body weight at "a slope of about 20–30 kcal/d per kg". Treat expenditure as a straight line in weight with that slope.

A person loses 8 kg. Work out how much their expenditure changes along the line, at each end of the range. Give the unit.

8. Table 5.2 of the FAO/WHO/UNU report (2004) estimates basal metabolic rate from body weight. For men aged 18 to 30 it is 15.057 times weight in kg, plus 692.2, in kcal a day.

Find dBMR/dW with its unit. Then work out BMR at 60 kg and at 70 kg, and check the difference against the derivative.

9. The same FAO/WHO/UNU report multiplies BMR by the physical activity level, PAL, to get total energy expenditure, TEE. Its worked example uses a PAL of 1.75.

Using the Table 5.2 line for men aged 18 to 30, BMR = 15.057W + 692.2, find dTEE/dW at a PAL of 1.75. Compare it with Hall and Guo's slope of "about 20–30 kcal/d per kg" for expenditure. Say what the comparison does and does not show.

10. A made-up weight curve is W(t) = 84 minus 0.15t + 0.0005 t^2, with W in kg and t in days.

Find W'(60) and W"(t), with units. Say what W is doing at day 60. Then find the day on which W' reaches zero.

11. Here is a worked answer. Find the step that broke.

> f(x) = 3 x^2 + 5
> d/dx (3 x^2) = 6x
> d/dx (5) = 5
> f'(x) = 6x + 5
> so f'(0) = 5

12. Here is a worked answer. Find the step that broke.

> g(t) = 2^t
> by the power rule, g'(t) = t times $2^{(t \text{ minus } 1)}$
> so g'(1) = 1 times 2^0 = 1

13. Here is a worked answer. Find the step that broke.

> h(t) = t^2 times t^3
> d/dt (t^2) = 2t and d/dt (t^3) = 3 t^2
> the derivative of a product is the product of the derivatives
> h'(t) = 2t times 3 t^2 = 6 t^3
> so h'(1) = 6

14. Here is a worked answer. Find the step that broke.

> a patient's weight follows W(t) = 95 minus 0.3t + 0.002 t^2 over 60 days
> W"(t) = 0.004, which is positive
> a positive derivative means rising
> so from day 1 the patient's weight has been going up

The curve is made up for this problem.

15. A colleague says this at a case meeting.

> *Each kilo she loses cuts her expenditure by 20 to 30 kcal a day. She has lost 10 kg. So she only needs to eat 200 to 300 kcal less than when she started to keep losing at the same speed.*

The patient is made up. The slope is Hall and Guo's (2017). Decide what to compute, compute it, and say what your answer does not establish.

16. A news report on a weight-loss medicine says this.

> *Patients lost weight quickly at first, but loss slowed after month 6, which suggests the medicine stopped working.*

The report is made up, and so are these monthly average weights for the group.

month	weight (kg)
0	100
1	97
2	94.5
3	92.5
4	91
5	90
6	89.3
7	88.8
8	88.5

Decide what to compute, compute it, and say what your answer does not establish.

4 · The number e, natural logarithms and change proportional to amount

DEFINITION. The number e is the base of the one exponential curve, $e^{(t)}$, whose slope where it crosses the vertical axis, at t = 0, is exactly 1. To five decimal places e = 2.71828.

The natural logarithm of a positive number x, written ln x, is the power to which e must be raised to give x. It is defined only for numbers above zero, and ln 1 = 0.

For constants A and k, the function $y = A\ e^{(kt)}$ has derivative $dy/dt = k\ A\ e^{(kt)}$, which is k times y.

Where the quantity shrinks, write $y = A\ e^{(-kt)}$ with k above zero. The time it takes to fall to half of any starting value is the half-time, ln 2 divided by k, which is about 0.693 divided by k. The time to fall to one twentieth, 5% of the start, is ln 20 divided by k, about 3.00 divided by k.

A quantity W that approaches a level L in this way follows W = L plus (W0 minus L) times $e^{(-kt)}$, where W0 is its value at t = 0. The gap W minus L is the shrinking exponential. Its rate of change is dW/dt = -k times (W minus L). [26, 27, 28, 29, 30, 31]

IN PLAIN TERMS. Book 0 built powers and logarithms on ten (Book 0, A6 and Book 0, A7). Papers on body weight use a different base, a number written e.

Take a base, say 2, and ask how steep the curve $2^{(t)}$ is where it starts, at t = 0. Take a very short step, 0.0001, and divide the change by the step.

```
2^0.0001 = 1.0000693
(1.0000693 minus 1) divided by 0.0001 = 0.693
```

The curve $2^{(t)}$ starts with a slope of 0.693. Do the same with 3 and you get 1.099. So somewhere between 2 and 3 is a base whose curve starts with a slope of exactly 1. That base is e, and it is 2.71828 to five decimal places. Your calculator has a key for it, usually marked $e^{(x)}$.

Take $y = e^{(kt)}$, where k is a fixed number, and find its rate at any time t. Step forward a short time h. Multiplying two powers of the same base adds their exponents (Book 0, A6), so read that rule backwards.

```
e^(kt plus kh) = e^(kt) times e^(kh)
the change = e^(kt) times e^(kh) minus e^(kt)
the change = e^(kt) times (e^(kh) minus 1)
the change divided by h = e^(kt) times (e^(kh) minus 1) divided by h
```

As h shrinks, ($e^{(kh)}$ minus 1) divided by h settles on k. Call kh by one letter, u. Then ($e^{(kh)}$ minus 1) divided by h is k times ($e^{(u)}$ minus 1) divided by u. And ($e^{(u)}$ minus 1) divided by u, for u close to zero, is the slope of $e^{(t)}$ at the start, which is 1.

So the rate of $e^{(kt)}$ is k times $e^{(kt)}$. Put a constant A in front, and by the constant-multiple rule (section 3) the rate of A $e^{(kt)}$ is k times A $e^{(kt)}$.

The rate is k times the amount itself. The number k is the rate constant, the rate divided by the amount. Its unit is the rate's unit divided by the amount's: (kilograms a day) divided by kilograms is per day. For a quantity that shrinks, written A $e^{(-kt)}$ with k above zero, the rate is -k times the amount, so the rate divided by the amount is -k.

A logarithm undoes a power. The natural logarithm, written ln, undoes a power of e. So $\ln(e^3) = 3$, $\ln e = 1$ and $\ln 1 = 0$.

It turns multiplication into addition, for the same reason $\log_{10}$ does.

```
ln(ab) = ln a plus ln b
ln 2 plus ln(1/2) = ln 1 = 0, so ln(1/2) = -ln 2
```

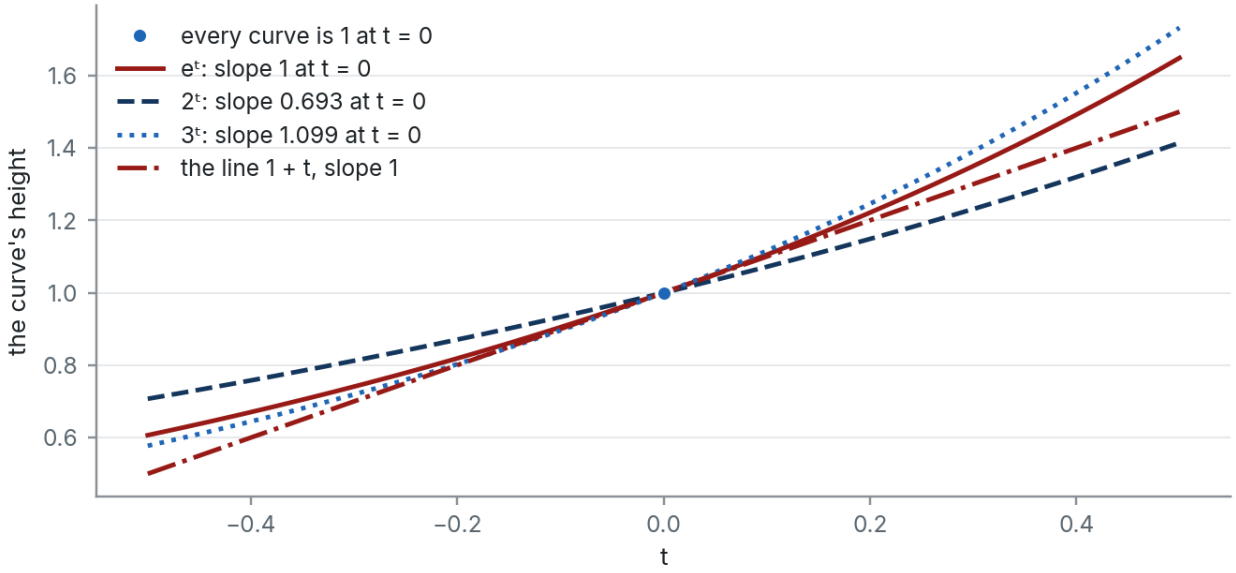
Now a quantity that shrinks, $y = A\, e^{(-kt)}$, with k above zero. When has it fallen to half? Set the curve equal to half its start, and divide both sides by A.

```
A e^(-kt) = A divided by 2
e^(-kt) = 1/2
```

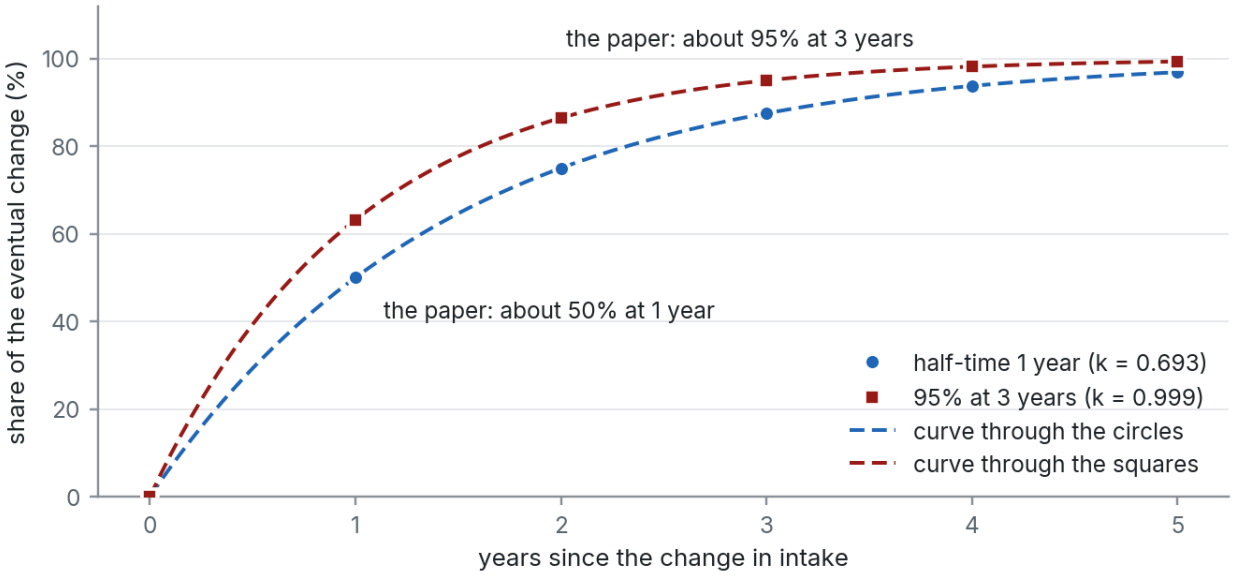
Both sides are the same number, so their natural logarithms are the same number.

```
-kt = ln(1/2) = -ln 2
t = ln 2 divided by k
```

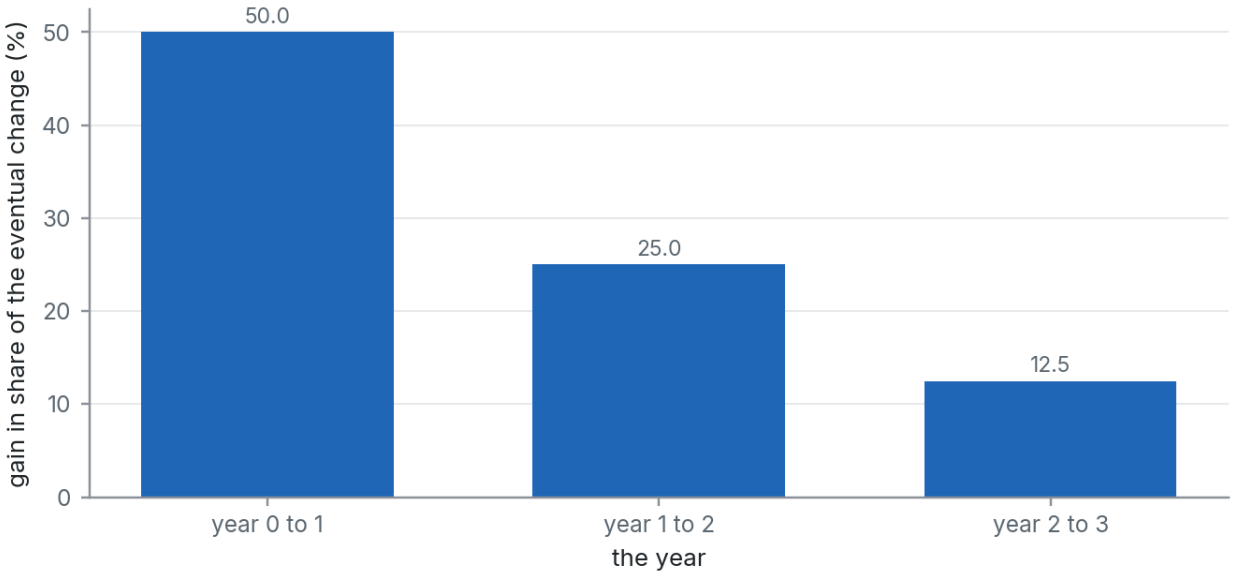
That time is the half-time. It does not depend on A, so the quantity halves in the same time from any starting point. Put one twentieth in place of one half and the same steps work. They give the time to fall to 5% of the start, which is 95% of the way to zero.



Three exponential curves near t = 0, where each passes through 1. $2^{(t)}$ starts with a slope of 0.693 and $3^{(t)}$ with 1.099. $e^{(t)}$, with e = 2.71828, starts with a slope of exactly 1, so it touches the straight line 1 + t there.



Two exponentials, each matching one of Hall's rounded figures. The one with a half-time of 1 year reaches 50% at 1 year and only 87.5% at 3. The one that reaches 95% at 3 years is already at 63.2% after 1. No single exponential passes through both figures.



The one-year half-time curve read once a year, 0.0, 50.0, 75.0 and 87.5, drawn as the gain in each year. Each gain is half the one before: 25.0 divided by 50.0 and 12.5 divided by 25.0 are both 0.5. That steady ratio is $e^{(-k)}$ for a step of one year, and gives back k = 0.693 per year.

ILLUSTRATION. Here is a claim you will meet, and a test it fails.

Hall and colleagues modelled a man weighing 100 kilograms who cuts his intake by 2 megajoules a day and keeps it cut (*Lancet* 2011). Their model has his weight level off at about 75 kilograms. It takes "roughly 1 year to reach half of the maximum weight loss", and reaches "95% of this value after about 3 years".

The tempting reading is that the weight closes its gap as one exponential with a half-time of one year. Test it. Start from the half-time and work out k.

```
0.693 divided by 1 = 0.693
```

So k is 0.693 per year. Now the time to 95%.

```
2.996 divided by 0.693 = 4.32
```

About 4.3 years, not 3.

Start from 95% at 3 years and work out k, then the half-time.

```
2.996 divided by 3 = 0.999
0.693 divided by 0.999 = 0.694
```

A half-time of 0.694 years, about 8 months, not 1 year.

Now put the two curves side by side. The change so far, divided by the whole change W0 minus L, is the share reached.

```
W0 minus W = (W0 minus L) times (1 minus e^(-kt))
(W0 minus W) divided by (W0 minus L) = 1 minus e^(-kt)
```

So the share of the eventual change reached by time t is 1 minus $e^{(-kt)}$. Work each row out on your calculator, times 100 for a percentage.

years	half-time 1 year (k = 0.693)	95% at 3 years (k = 0.999)	the paper's rounded figures
0	0.0	0.0	
1	50.0	63.2	about 50
2	75.0	86.4	
3	87.5	95.0	about 95
4	93.7	98.2	
5	96.9	99.3	

Neither column passes through both figures. One exponential cannot give both, because for one exponential the time to 95% is always 4.32 half-times. The paper's figures give 3 divided by 1, which is 3.

Book 0 gave you a test for a shape that flattens against a ceiling (Book 0, C6). Take values at equal steps, subtract each from the next, then divide each difference by the one before. A ratio that holds steady below one means a ceiling you can work out.

Take the first column of the table above, the curve with a one-year half-time. Read it once a year: 0.0, 50.0, 75.0, 87.5. Subtract each from the next.

```
50.0 minus 0.0 = 50.0
75.0 minus 50.0 = 25.0
87.5 minus 75.0 = 12.5
```

Now divide each difference by the one before.

```
25.0 divided by 50.0 = 0.5
12.5 divided by 25.0 = 0.5
```

A steady ratio of 0.5.

The ratio is what the gap to the ceiling, and each year's gain, is multiplied by over one step, so it is $e^{(-k)}$ for a step of one year. Take ln of both sides to get k back.

```
e^(-k) = 0.5
-k = ln 0.5 = -0.693
k = 0.693 per year
```

That is the k you started from. Divide minus ln of the ratio by the length of the step.

Figures quoted in this illustration are from [32].

WHERE THIS PICTURE BREAKS. Both columns are what one exponential would give. Neither is the model's own curve, which the paper does not print as a formula. Nothing here says a real person follows either column.

Rounded or noisy readings make the ratios wobble, and one wobbling ratio is not evidence of any shape.

MUST KNOW POINTS FOR YOU.

- For any single exponential, the time to 95% of the way is 4.32 half-times, because ln 20 divided by ln 2 is 4.32. Hall's "about 1 year" and "about 3 years" are in another ratio. Do not fill in the months between with an exponential you chose yourself.
- Use ln, not $\log_{10}$, in a half-time. The calculator key marked log is $\log_{10}$, and $\log_{10} 2$ divided by k gives 0.301 divided by k, less than half the right answer. The rate constant k was defined with e, so only ln undoes it.
- Write the unit beside k every time. A k of 0.693 per year used with t in days gives an answer 365 times too fast.
- To test whether a claimed curve is exponential, divide its rate by its value, or, for a curve levelling off at L, by its gap to L, at two or more times. With readings at equal steps, divide each difference by the one before; the ratio must hold steady.
- A gap that closes exponentially closes fastest at the start and slowest at the end. When a claim calls weight loss "exponential", ask whether it means growth or a closing gap, and for its k with a unit.
- A patient may ask how long a change in intake takes to show in full. Give the model's figures as rough figures. Roughly a year to half, and about three years to most of it, for the model's average adult (Hall 2011). Do not give month-by-month figures from one exponential.

EXERCISE 1 (calculation). A quantity falls as $y = A\,e^{(-kt)}$, with k above zero. Derive the time at which it has fallen to 5% of its starting value. Write the move you make beside every line, as a word or two: for example “divide both sides by A”. Then say why the answer does not depend on A.

PRACTICE. Sixteen problems on this technique, easiest first. Work them on paper before turning to the answers, which are in the appendix at the back under this section’s name.

1. Use the $e^{(x)}$ and ln keys of your calculator. Work out each of these to three decimal places: e^{1}, e^{2}, $e^{(-1)}$, ln 1, ln 10 and $\ln(e^{3})$.

2. Show with numbers that ln 20 equals ln 4 plus ln 5. Then show that ln(1/2) equals minus ln 2. Work to three decimal places.

3. A quantity follows $y = 40\,e^{(0.05t)}$. Work out its rate of change, dy/dt. Then work out y, dy/dt and dy/dt divided by y at t = 0 and at t = 10.

4. A quantity follows $y = 200\,e^{(-0.1t)}$. Work out its half-time and the time it takes to fall to 5% of its starting value. Then work out y at each of those two times.

5. A quantity falls exponentially with a half-time of 14 days. Work out its rate constant k. Then work out the share of the starting value left after 28 days, and after 10.

6. Solve $100\,e^{(-0.2t)} = 30$ for t.

7. A quantity is read at equal steps of 2 units of time: 60, 70, 75, 77.5. Find the level it is approaching, its rate constant k, and its half-time.

8. Hall and colleagues (*Lancet* 2011) give a rule of thumb for an average adult with overweight. It reads: “every change of energy intake of 100 kJ per day will lead to an eventual bodyweight change of about 1 kg”. Half of that change comes “in about 1 year”.

A person cuts intake by 500 kJ a day and holds it. Work out the eventual change the rule of thumb gives. Then suppose the change comes as one exponential with a half-time of exactly 1 year. Work out how much of it would have come by 6 months and by 2 years. Say which of your figures are the paper’s.

9. Hall and colleagues’ modelled man starts at 100 kilograms and levels off at about 75 kilograms (*Lancet* 2011). Suppose his weight closes the gap as one exponential that is 95% of the way there at 3 years. Work out his weight at 1 year. Then work it out again for an exponential with a half-time of 1 year, and give the difference.

10. Take the same man, with a gap of 25 kilograms, and suppose it closes as one exponential with a half-time of 1 year. Work out the rate of change of his weight at the very start, in kilograms a year and in grams a day. Then compare it with his average rate over the first year.

11. Here is a worked answer. Find the step that broke.

```
a quantity falls exponentially with k = 0.1 per day
half-time = log10 2 divided by k
log10 2 = 0.301
0.301 divided by 0.1 = 3.01
so it halves every 3.01 days
```

12. Here is a worked answer. Find the step that broke.

```
a change reaches its eventual size with k = 0.693 per year
95% of the way means e^(-kt) = 0.95
-kt = ln 0.95 = -0.0513
t = 0.0513 divided by 0.693 = 0.074
so 95% of the change comes in 0.074 years, about 27 days
```

13. Here is a worked answer. Find the step that broke.

```
y = 40 e^(0.05t)
by the power rule, bring the power down and take one off it
dy/dt = 40 times 0.05t times e^(0.05t minus 1)
at t = 0, dy/dt = 0
so the curve is flat where it starts
```

14. Here is a worked answer. Find the step that broke.

```
a gap closes exponentially with a half-time of 1 year, so k = 0.693
after 30 days the share left is e^(-0.693 times 30)
e^(-20.79) = 0.000000001
so the change is complete within a month
```

15. A colleague reads the rule of thumb in Hall and colleagues (*Lancet* 2011). It has "half of the weight change being achieved in about 1 year and 95% of the weight change in about 3 years". The colleague says this.

> *That is an exponential with a one-year half-time. After three years it would be only 87.5% of the way, so the paper contradicts itself.*

Decide what to compute, compute it, and say what your answer does not establish.

16. A health magazine prints this line. The patient and the 4 kilograms are made up for this problem.

> *Weight loss slows exponentially. So a patient who has lost 4 kg in the first three months will have lost 16 kg by the end of the year.*

Decide what to compute, compute it, and say what your answer does not establish.

5 · The integral of a rate is the change in the stock

DEFINITION. Let f(t) be a rate that depends on time t. The definite integral of f from a to b is written ∫ from a to b of f(t) dt. Cover the time from a to b with thin intervals of width Δt. Over each, draw a rectangle whose height is the rate in that interval. The integral is the limit of the sum of the rectangles' areas as the widths shrink towards zero.

If f is the rate of change of a quantity Q, then ∫ from a to b of f(t) dt equals Q(b) minus Q(a). This is the net change theorem: the integral of a rate over an interval is the change in the stock over that interval.

Two cases can be done exactly with no limit. In both, every slicing gives the same total, so letting the slices shrink changes nothing. A constant rate c gives a rectangle, c times (b minus a). A rate that changes along a straight line gives a trapezium, the average of the two end rates times (b minus a).

The average value of f from a to b is the integral divided by (b minus a).

Applied to a body, with stored energy ES, energy intake EI and energy expenditure EE, the identity of Book 1, section 5 written as rates is dES/dt = EI minus EE. So ES(b) minus ES(a) = ∫ from a to b of (EI minus EE) dt. [33, 34, 35, 36, 37, 38, 39]

IN PLAIN TERMS. Here is how a paper writes an integral, the total under a rate line that Book 0 drew as rectangles (Book 0, C8).

```
∫ from a to b of f(t) dt
```

The ∫ is a stretched S, for sum. The a at its foot is where the time starts, and the b at its top is where it stops. Those two are the limits of integration.

Next comes f(t), the rate being added up. Its name is the integrand. Last, dt says the adding runs over time t.

It means this. Cut the time from a to b into thin slices. Multiply the rate in each slice by its width. Add them all. Then let the slices get thinner, until the total stops changing.

Here a rate can drop below zero, because a stock can lose as well as gain. A rectangle below the time axis has a negative height, so its area counts as negative. Area below the axis is subtracted. The result is called the net signed area.

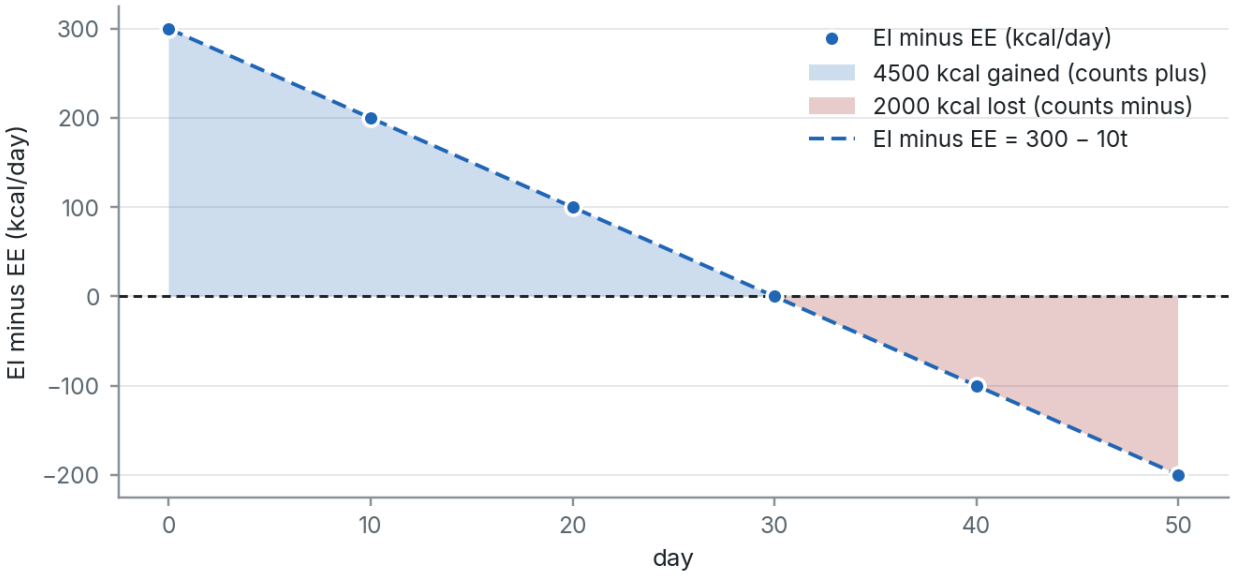
The integral of a rate over an interval equals the change in the stock over that interval. This is called the net change theorem.

```
the stock at b minus the stock at a = ∫ from a to b of the rate dt
```

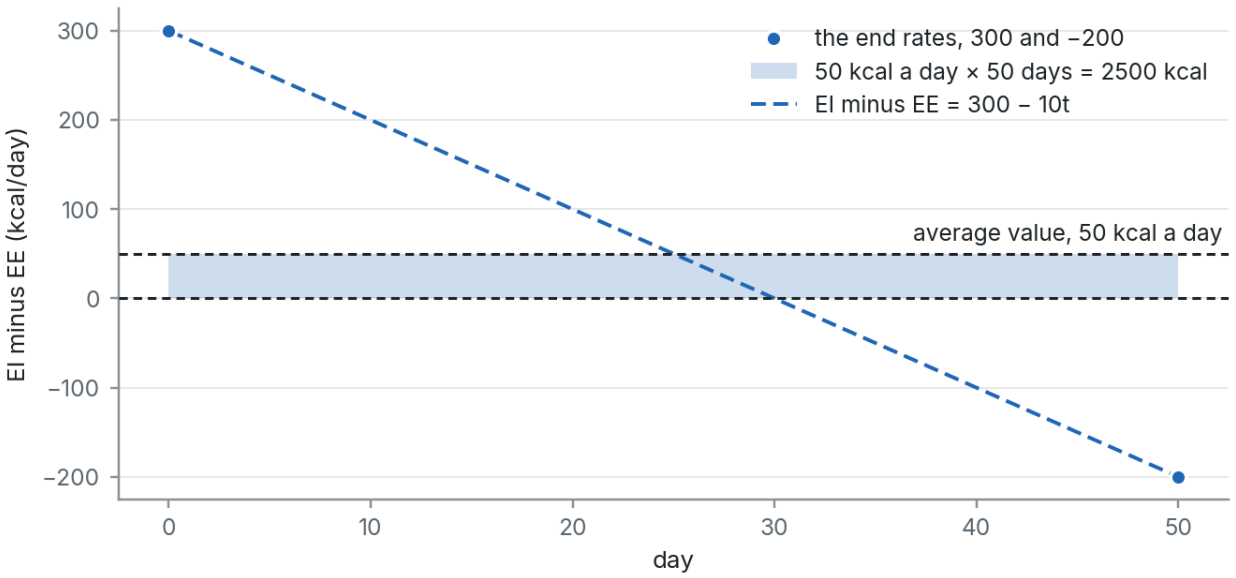
Run the time backwards, from b to a, and the same change is undone. So the integral from b to a is minus the integral from a to b.

For a body's energy stores, the rate is intake minus expenditure (Book 1, section 5). Write the stores as ES, intake as EI and expenditure as EE.

```
dES/dt = EI minus EE
ES(b) minus ES(a) = ∫ from a to b of (EI minus EE) dt
```



A made-up net rate falling in a straight line from 300 to minus 200 kilocalories a day over 50 days, crossing zero at day 30. The area above the axis is 4500 kilocalories gained, the area below is 2000 lost, and the change in stores is the net signed area, 4500 minus 2000 = 2500.



The same net rate with its average value, 50 kilocalories a day, the mean of the end rates 300 and minus 200. A rectangle 50 high and 50 days wide has area 2500 kilocalories, the same as the net signed area under the line: that is the trapezium's one-line way.

ILLUSTRATION. Take a made-up case, with numbers chosen so that the rate is a straight line. A person's net rate, intake minus expenditure, is 300 kilocalories a day on day 0. It falls by 10 kilocalories a day, every day, for 50 days.

day	EI minus EE (kcal/day)
0	300
10	200
20	100
30	0
40	-100
50	-200

A trainee is asked how much the person's energy stores changed over the 50 days. They draw the line and shade everything between it and the time axis. There are two triangles, and a triangle's area is half its base times its height. They work out both and add them.

```
0.5 times 30 times 300 = 4500
0.5 times 20 times 200 = 2000
4500 plus 2000 = 6500
```

They report a gain of 6,500 kilocalories. Find the flaw before you read on.

From day 30 to day 50 the rate is below zero. The 2,000 kilocalories under the axis is energy taken out, not put in. It has to be subtracted.

```
4500 minus 2000 = 2500
```

The stores rose by 2,500 kilocalories over the 50 days.

Now the one-line way. The rate is a straight line, so the whole shape counts as one trapezium. Take the average of the two end rates, keeping the minus sign, and multiply by the time.

```
300 plus -200 = 100
100 divided by 2 = 50
50 times 50 = 2500
```

The same 2,500 kilocalories. The middle number, 50 kilocalories a day, is the average value of the rate.

The rate is 300 minus 10t, with t in days.

```
∫ from 0 to 50 of (300 minus 10t) dt = 2500
```

Check its unit. It is kilocalories a day times days, which is kilocalories.

Hall and colleagues' consensus statement (*American Journal of Clinical Nutrition* 2012) writes it as ES = EI minus EO. EO is the expenditure this book writes EE (section 1). So in their equation ES is the rate of change of the stores, not the stores.

WHERE THIS PICTURE BREAKS. The straight-line rate is made up. A real person's net rate does not change in a straight line and is never measured at every moment.

Stored energy is not body weight. Turning 2,500 kilocalories into kilograms needs to know what tissue was gained or lost (Book 1, section 2).

MUST KNOW POINTS FOR YOU.

- Area below the time axis is subtracted. Before you accept a change in stores, check that the days of loss were taken away.
- A shrinking surplus is still a surplus. While intake minus expenditure stays above zero, the stores rise, however fast the surplus is shrinking.
- The trapezium is exact only for a rate that changes in a straight line between the two readings. For any other rate it is an estimate.
- Hall's ES = EI minus EO is an equation between rates. Before you quote any figure called ES, ask whether its unit is kilocalories or kilocalories a day.
- Check the unit of every integral: the rate's unit times the time's unit.
- A small error in a daily rate grows in the integral. Hall and colleagues put the uncertainty of doubly labelled water at more than 100 kilocalories a day.

EXERCISE 1 (interpretation). Hall and colleagues' consensus statement (*American Journal of Clinical Nutrition* 2012) says this about the energy balance equation.

> *the rate of change in body ES is equal to the difference between the rates of EI and EO.*

They add that every term is energy per unit of time.

Write the sentence first as an equation between rates, in this book's notation. Then write it as an integral that gives the change in stores between day a and day b. Say what each symbol stands for and give its unit.

PRACTICE. Fourteen problems on this technique, easiest first. Work them on paper before turning to the answers, which are in the appendix at the back under this section's name.

1. Look at ∫ from 2 to 7 of (3t plus 1) dt. Name the lower limit, the upper limit, the integrand and the variable. Then work out the integral.

2. Work out ∫ from 0 to 6 of 4 dt, and ∫ from 0 to 6 of -4 dt.

3. Work out ∫ from 0 to 8 of 5t dt.

4. Work out ∫ from 0 to 10 of (6 minus t) dt. Then work out the total area between the line and the axis, counting all of it as positive, and say why the two differ.

5. Work out the average value of 2t plus 4 from t = 1 to t = 5.

6. A rate f(t) has ∫ from 0 to 10 of f(t) dt = 50 and ∫ from 0 to 4 of f(t) dt = 18. Work out ∫ from 4 to 10 of f(t) dt, and ∫ from 10 to 4 of f(t) dt.

7. Doubly labelled water measures what a person spends (Book 1, section 3). Hall and colleagues put the uncertainty of its figure for expenditure at more than 100 kilocalories a day (*American Journal of Clinical Nutrition* 2012). Take a measuring period of 14 days. That is the upper end of the range in Book 1, section 3. Work out what that uncertainty becomes in the total spent over the 14 days.

8. Hall and colleagues (*Lancet* 2011) model a cut in intake of 480 kilocalories a day. First suppose the person's net deficit stays at 480 a day for 365 days. Then suppose instead that it falls in a straight line from 480 to 240 a day over the same 365 days. That second shape is made up for this problem. Work out the total deficit each way, and the average daily deficit in the second case.

9. A person's energy stores fall by 9,000 kilocalories over 30 days. The figures are made up for this problem. Work out their average net rate. Then suppose the rate changed along a straight line and started at minus 500 kilocalories a day. Work out where it ended.

10. Here is a worked answer. Find the step that broke.

```
the net rate starts at -400 kcal/day and rises in a straight line to 200 kcal/day
over 30 days
it crosses zero at day 20
area below the axis = 0.5 times 20 times 400 = 4000
area above the axis = 0.5 times 10 times 200 = 1000
total = 4000 plus 1000 = 5000
so the stores fell by 5000 kcal
```

11. Here is a worked answer. Find the step that broke.

```
a surplus of 300 kcal/day is held for 4 weeks
the change in stores = ∫ from 0 to 4 of 300 dt
300 times 4 = 1200
so the stores rose by 1200 kcal
```

12. Here is a worked answer. Find the step that broke.

```
the net deficit was read three times: 400 kcal/day on day 0, 300 on day 10, 0 on day
40
take the rate as a straight line between readings
average rate = (400 plus 300 plus 0) divided by 3 = 233.3
total = 233.3 times 40 = 9333
```

13. A trial report says this. The trial and its figures are made up for this problem.

> *Participants' average energy imbalance was minus 150 kcal a day over 12 weeks, so each lost 12,600 kcal from their stores.*

Hall and colleagues (*American Journal of Clinical Nutrition* 2012) say the combined error in measuring energy imbalance "can easily reach 1000 kcal/d".

Decide what to compute, compute it, and say what your answer does not establish.

14. A colleague says this about a patient. The patient and the figures are made up for this problem.

> *Her surplus has been falling for two months, from 400 kcal a day to zero. So she has been losing weight.*

Decide what to compute, compute it, and say what your answer does not establish.

6 · Undoing a derivative: antiderivatives and the fundamental theorem

DEFINITION. An antiderivative of a function f is a function F whose derivative is f, so that dF/dt = f(t).

Each of these is checked by differentiating back, with the rules of section 3 and section 4:

the function f(t)	an antiderivative F(t)
a constant c	c times t
$t^{(n)}$, for n a whole number	$t^{(n \text{ plus } 1)}$ divided by (n plus 1)
$e^{(kt)}$, for k not zero	$e^{(kt)}$ divided by k

Sums and constant multiples are done term by term.

The fundamental theorem of calculus, in the part called the evaluation theorem, says this. If f is continuous from a to b (its graph has no jumps; every rate in this book is) and F is any antiderivative of f, then ∫ from a to b of f(t) dt = F(b) minus F(a). [40, 41, 42, 43]

IN PLAIN TERMS. section 5 worked out integrals from shapes: rectangles, triangles, trapeziums. That works only for a rate that is steady or changes in a straight line. Most rates bend. This section gives a way that does not need a shape.

Given a rate, find a function whose rate it is. That function is called an antiderivative.

Say the rate is 2t. Which function has the rate 2t? The power rule says t^2 does. So t^2 is an antiderivative of 2t.

So is t^2 plus 5, and t^2 plus any fixed number. A fixed number has a rate of zero, so adding one changes nothing about the rate. That is why an antiderivative is written with plus C on the end, where C is any constant.

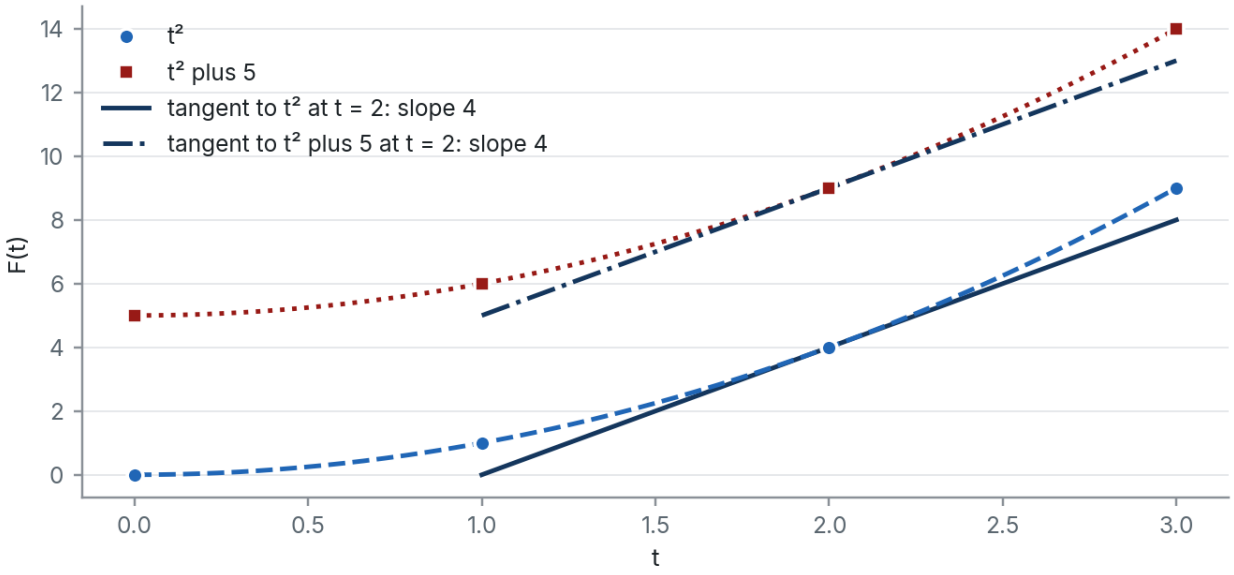
You never have to trust an antiderivative. Differentiate it and see whether the rate comes back.

the rate	an antiderivative
a steady rate c	c times t
$t^{(n)}$, n a whole number	$t^{(n \text{ plus } 1)}$ divided by (n plus 1)
$e^{(kt)}$	$e^{(kt)}$ divided by k

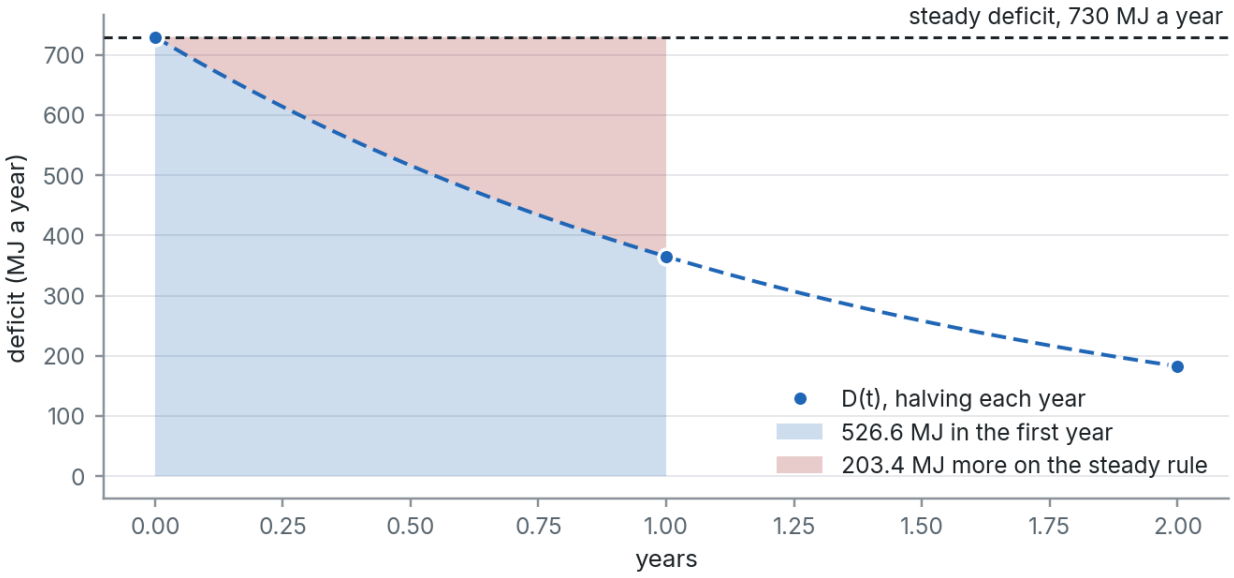
The fundamental theorem of calculus says an integral is the antiderivative at the end minus the antiderivative at the start.

```
∫ from a to b of f(t) dt = F(b) minus F(a)
```

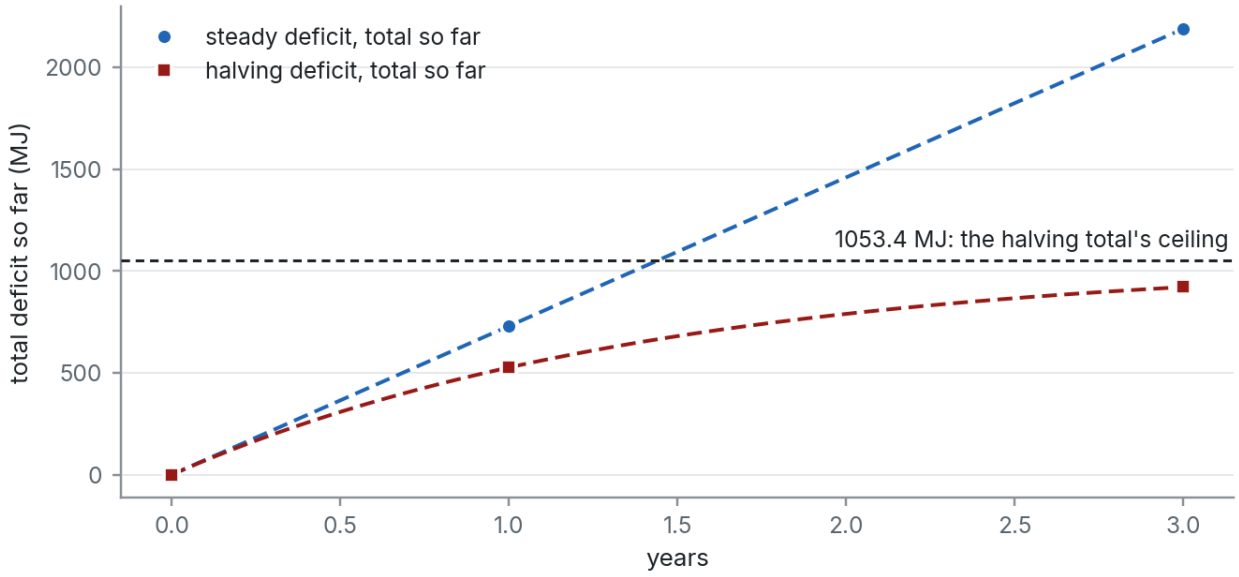
Here F is any antiderivative of f. The C cancels, because it is added at both ends and then subtracted. So leave it out when you work out an integral.



Two antiderivatives of the rate 2t: t^2, and t^2 plus 5. One is the other lifted by 5 at every t, so they rise equally fast. At t = 2 both tangents have slope 4, which is 2t there. Adding a constant changes the level, never the rate.



The made-up deficit D(t) = 730 $e^{(-0.693t)}$ megajoules a year, which halves each year, against a steady 730. Over the first year the area under the halving deficit is F(1) minus F(0) = 526.6 megajoules. The steady rectangle is 730, and the strip between them, 203.4, is what the steady rule over-counts.



Totals so far. A steady deficit of 730 megajoules a year adds up in a straight line: 730 after 1 year, 2190 after 3. The halving deficit adds up to 526.6 in the first year and 921.7 by year 3, and levels off under 1053.4 megajoules, which it never passes.

ILLUSTRATION. Hall and colleagues model a man who cuts his intake by 2 megajoules a day, which is 480 kilocalories a day (*Lancet* 2011). Keep the deficit at 2 megajoules a day, as the static rule of Book 1, section 7 does. Then one year of it is a rectangle.

```
2 times 365 = 730
```

So a deficit of 2 megajoules a day is 730 megajoules a year.

His expenditure falls as his weight falls, so his real deficit shrinks. Suppose, to see the arithmetic, that it halves every year. This shape is made up here, and it is not the paper's model. A half-time of 1 year means k = 0.693 per year (section 4). With t in years, the deficit in megajoules a year is this.

$$D(t) = 730\ e^{(-0.693t)}$$

The rate of 730 $e^{(-0.693t)}$ is -0.693 times 730 $e^{(-0.693t)}$. That is not the deficit. It is -0.693 times the deficit. So divide by -0.693 to cancel that factor.

$$730 \text{ divided by } 0.693 = 1053.4$$
$$F(t) = -1053.4\ e^{(-0.693t)}$$

Check it: differentiate F and see whether 730 $e^{(-0.693t)}$ comes back. The rate of F is -0.693 times -1053.4 $e^{(-0.693t)}$, which is 730 $e^{(-0.693t)}$. Now work out F(1) minus F(0) on your calculator. $e^{(-0.693)}$ is 0.5001, and e^{0} is 1.

```
F(1) minus F(0) = -1053.4 times 0.5001 plus 1053.4
1053.4 times (1 minus 0.5001) = 1053.4 times 0.4999
1053.4 times 0.4999 = 526.6
```

About 526.6 megajoules in the first year, against 730 for the steady deficit.

Take the total up to any later time T. It is F(T) minus F(0), which is 1053.4 times (1 minus $e^{(-0.693T)}$). As T grows, $e^{(-0.693T)}$ shrinks towards zero, so the total creeps up towards 1053.4 megajoules and never passes it.

A deficit that shrinks adds up to less.

Figures quoted in this illustration are from [32].

WHERE THIS PICTURE BREAKS. Every total here is energy, in megajoules. Turning it into kilograms needs to know what tissue was lost (Book 1, section 2).

MUST KNOW POINTS FOR YOU.

- Check every antiderivative by differentiating it. If the rate you started with does not come back, the antiderivative is wrong, however reasonable it looks. It takes one line.
- The antiderivative of $e^{(kt)}$ is $e^{(kt)}$ divided by k, not $e^{(kt)}$. Dropping the k gives the wrong size. For a shrinking curve, $e^{(-kt)}$, dropping the k also gives the wrong sign: the antiderivative is $e^{(-kt)}$ divided by -k.
- "The deficit never quite reaches zero, so the loss keeps going for ever" is wrong as arithmetic. A deficit D0 $e^{(-kt)}$, where D0 is its value at the start, adds up to at most D0 divided by k, however long it runs.
- These rules cover steady rates, whole-number powers of t, $e^{(kt)}$, and sums and multiples of them. When no antiderivative is at hand, go back to rectangles and bound the error as Book 0, C8 does.
- A figure made by multiplying a daily deficit by 365, and then by a number of years, is a steady-deficit total. Any deficit that shrinks adds up to less. In a committee, call such a figure an upper limit, reached only if the deficit never shrinks.
- Take F at the upper limit minus F at the lower limit, in that order. The reverse gives the right size with the wrong sign.

EXERCISE 1 (calculation). A deficit shrinks as $D(t) = D0\ e^{(-kt)}$, where D0 is its size at t = 0 and k is above zero. Derive the total from t = 0 to t = T. Write beside each line the move you made. Then say what the total approaches as T grows, and why.

PRACTICE. Fourteen problems on this technique, easiest first. Work them on paper before turning to the answers, which are in the appendix at the back under this section's name.

1. Write an antiderivative of each: 6, 4t, $3\ t^2$, and t^3 plus 2t. Check each by differentiating it.

2. Work out $\int$ from 1 to 3 of $3\ t^2\ dt$.

3. Work out $\int$ from 0 to 2 of $e^{(3t)}\ dt$, to two decimal places.

4. Work out $\int$ from 0 to 4 of $10\ e^{(-0.5t)}\ dt$, to two decimal places.

5. Work out $\int$ from 2 to 5 of 2t dt twice: once with the antiderivative t^2, and once with t^2 plus 5. Say what happened to the 5.

6. A rate is 10 $e^{(-0.5t)}$. Work out its total from t = 0 onwards, over all time. Then work out how long it takes to add up to 95% of that.

7. Hall and colleagues (*Lancet* 2011) model a cut in intake of 480 kilocalories a day. Suppose a person's net deficit starts at 480 and falls by 0.5 kilocalories a day every day. That fall is made up for this problem. Use an antiderivative to work out the total deficit over 200 days. Then check it with the trapezium.

8. Take the same deficit of 480 kilocalories a day (*Lancet* 2011). Over the first 90 days, compare two totals. In the first, the deficit holds steady. In the second, it halves every 365 days, which is made up for this problem. Give the second as a share of the first.

9. A deficit shrinks as D0 $e^{(-0.693t)}$, with t in years, so it halves every year. Over the first year it adds up to 400 megajoules. Work out D0, in megajoules a year and then in megajoules a day.

10. Here is a worked answer. Find the step that broke.

> ∫ from 0 to 3 of (12 minus 2t) dt
> an antiderivative is F(t) = 12t minus t^2
> the integral = F(0) minus F(3)
> F(0) = 0 and F(3) = 36 minus 9 = 27
> 0 minus 27 = -27

11. Here is a worked answer. Find the step that broke.

> ∫ from 0 to 3 of t^2 dt
> raise the power by one: an antiderivative is t^3
> 3^3 minus 0^3 = 27

12. Here is a worked answer. Find the step that broke.

> a deficit of 480 kcal/day halves every year, so k = 0.693
> the total over the first 30 days = 480 times (1 minus $e^{(-0.693 \text{ times } 30)}$) divided by 0.693
> $e^{(-20.79)}$ is almost 0
> 480 divided by 0.693 = 692.6
> so the deficit adds up to about 693 kcal over the month

13. A patient information leaflet says this. The wording is made up for this problem.

> *Your deficit gets smaller as you lose weight, but it never reaches zero. So the weight loss keeps going for ever, and in the end you lose what the old calorie rule said you would.*

Decide what to compute, compute it, and say what your answer does not establish.

14. A journalist tells you this.

> *A modelling paper says the old rule overstates first-year weight loss by about 100%. So to correct any figure from the old rule, just halve it.*

Hall and colleagues (*Lancet* 2011) do say the static rule's first-year loss is "about 100% greater" than their model's. Decide what to compute, compute it, and say what your answer does not establish.

7 · A differential equation: a rule that gives the rate from the state

DEFINITION. A differential equation is an equation involving an unknown function and one or more of its derivatives. The ones in this book are first-order: each gives the rate of change of a quantity at a moment as a rule in its state, in the time, or in both. For body weight W in kilograms and time t in days, the form is dW/dt = f(W, t).

Such an equation has many solutions. An initial value, the state at one stated time, picks out one of them. The equation and the initial value together are an initial-value problem, and a function solves it only if it satisfies both.

With expenditure a straight line in weight, rising by epsilon, written ε, kilocalories a day for each kilogram, and a step change ΔEI in intake from a steady starting weight W0, the equation is:

```
ρ × dW/dt = ΔEI - ε × (W - W0),    with W = W0 at t = 0
```

This is Equation 1 of Polidori and colleagues (2016) with its physical activity and urinary glucose terms set to zero, rearranged.

A forward step, which is Euler's method taken one step at a time, approximates a solution. From the state at time t, compute the rate from the equation. Then take the state at t + h to be about W(t) + h × (the rate at t). Each step holds the rate fixed across one interval of length h, as one rectangle does in a running total. [44, 45, 46, 47, 37, 48, 49, 50, 51, 52, 53, 54, 55, 56]

IN PLAIN TERMS. Over a day, the change in the body's stored energy is what came in minus what went out.

```
dES/dt = EI - EE
```

ES is the energy stored in the body, in kilocalories. EI is energy intake and EE is energy expenditure, both in kilocalories a day.

Say each kilogram of weight gained or lost carries a fixed amount of energy. Call it rho, written ρ, in kilocalories per kilogram.

```
ρ × dW/dt = EI - EE
```

That line is true whenever ρ is fixed. The line above it is always true. Neither predicts anything. A smaller body spends less, so EE depends on the weight itself.

So write EE as a rule in W. It says each kilogram of weight adds a fixed number of kilocalories a day to expenditure. Call that number epsilon, written ε.

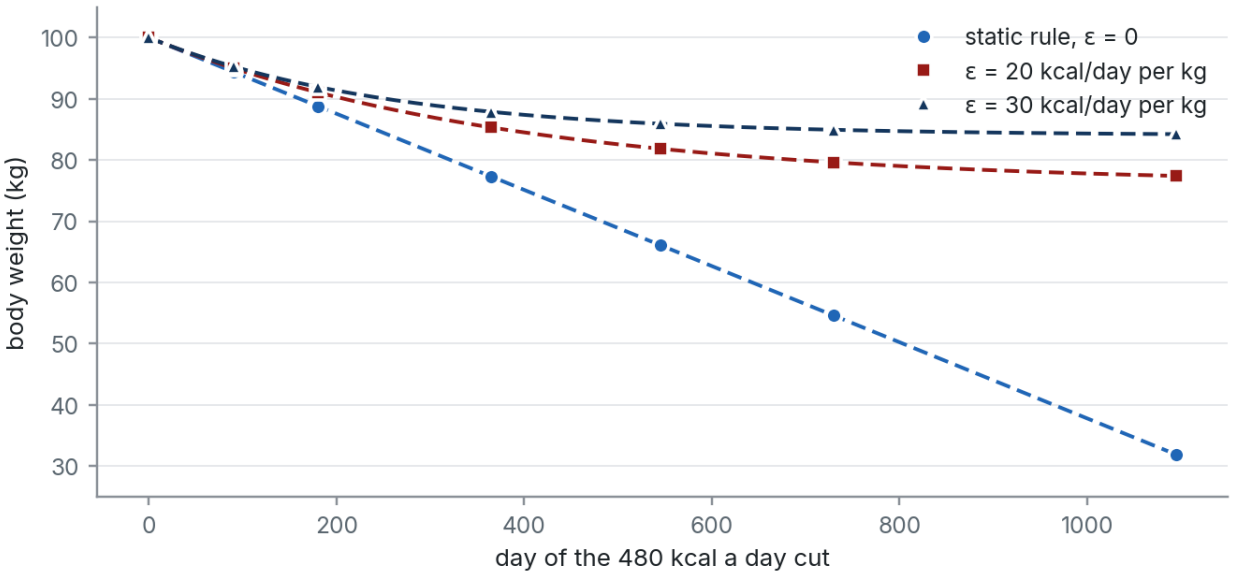
```
EE = EE0 + ε × (W - W0)
EI = EI0 + ΔEI
EI0 = EE0, because the weight was steady at the start
so ρ × dW/dt = EI - EE = ΔEI - ε × (W - W0)
```

EE0 and EI0 are expenditure and intake at the start, when the weight was W0. ΔEI is the change in intake.

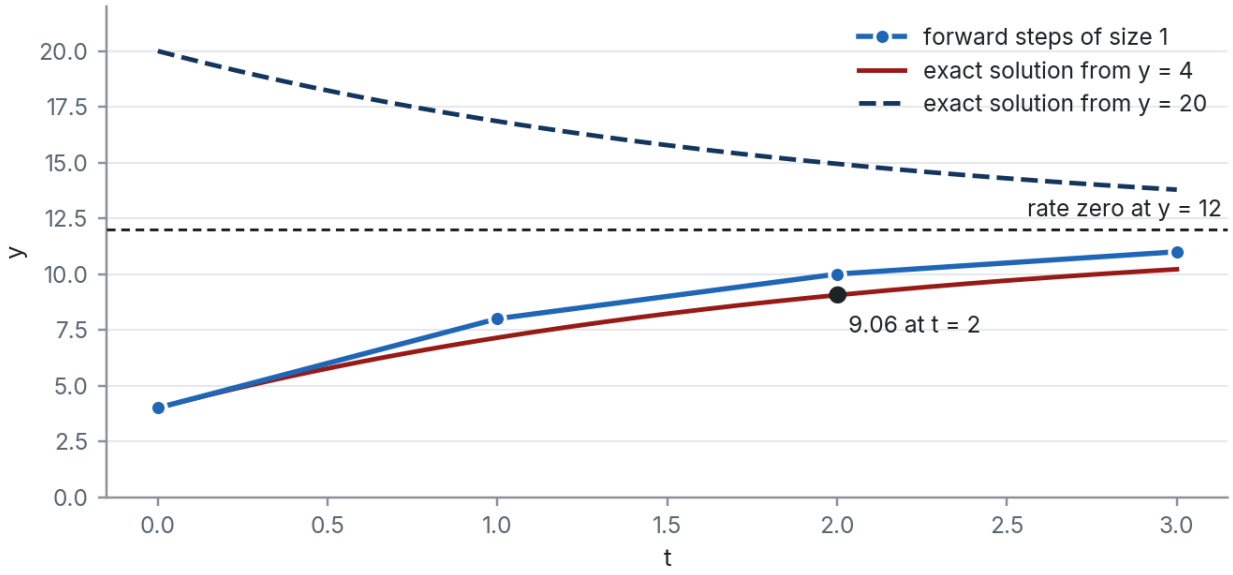
That is what a differential equation is: a rule that gives the rate from the state.

- An answer to it is not a number. It is a whole curve of weight against time. You check a curve by putting it into the equation. The two sides have to agree at every moment, not only at the start.
- Many curves obey the same rule. The starting weight picks out the one that is yours.

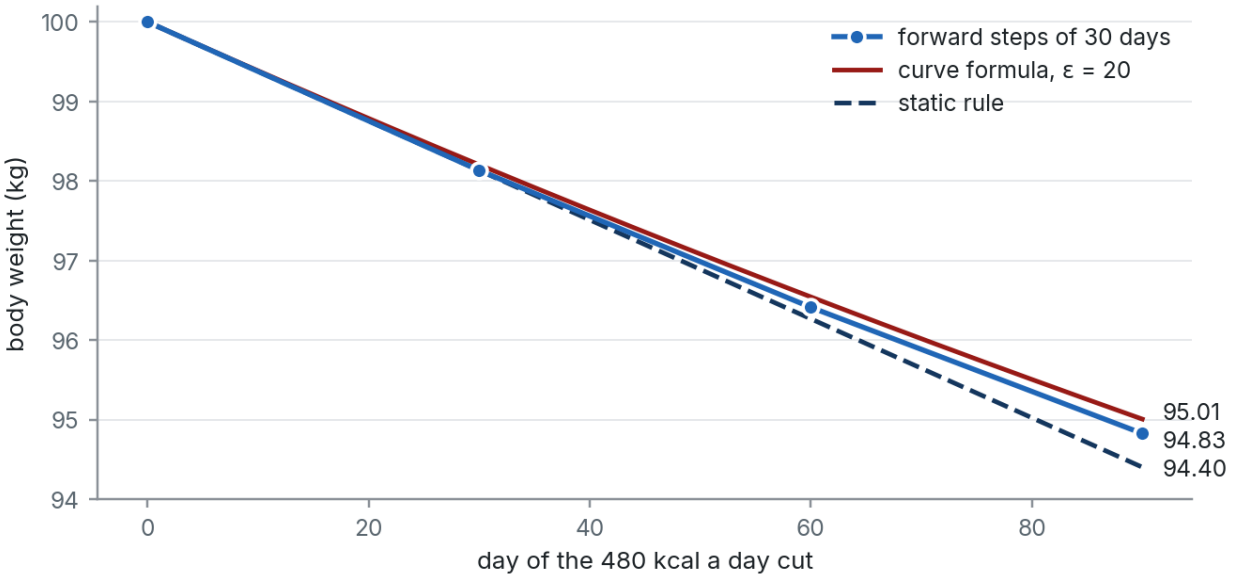
You can also walk the curve without a formula. Work out today's rate from today's weight. Move one day at that rate. Work out the new rate from the new weight. Move again.



Hall's 100 kg man cutting 480 kcal a day, from one equation with ρ = 7,716.18: the static rule (ε = 0) falls in a straight line to 31.88 kg by day 1095. With ε = 20 or 30 the loss slows, to 77.40 and 84.23 kg.



dy/dt = 6 − 0.5 × y. Forward steps of size 1 from y = 4 reach 8, 10 and 11, running ahead of the exact solution $12 - 8 \times e^{(-0.5t)}$, which is 9.06 at t = 2. Started at 20, the exact solution falls instead. Both close on 12, where the rate is zero.



The first 90 days of the 480 kcal a day cut, with ε = 20. Thirty-day forward steps reach 94.83 kg at day 90, a little below the curve formula's 95.01. The static rule's straight line is lower again, at 94.40.

ILLUSTRATION. Go back to Hall and colleagues' man from Book 1. He weighs 100 kilograms and cuts his energy intake by 2 megajoules a day. The paper converts that for you: "2 MJ per day (480 kcal per day)". So ΔEI, the change in intake, is minus 480 kilocalories a day.

Use 7,716.18 kilocalories per kilogram, the static rule's own figure from Book 1.

For ε, Hall and Guo (2017) give expenditure rising with weight "with a slope of about 20–30 kcal/d per kg". Take the lower end, 20 kilocalories a day for each kilogram. The man's equation is then:

```
7716.18 × dW/dt = -480 - 20 × (W - 100),    with W = 100 at t = 0
```

On the right, the cut in intake, plus the fall in expenditure since the start. When W is below 100, W − 100 is negative. So minus 20 times it is positive, and it eats into the deficit.

Start with the curve that fails. The static rule says his weight falls in a straight line.

```
480 divided by 7716.18 = 0.06221
W(t) = 100 - 0.06221 × t
```

Put that line into the equation a year in, at t = 365. The line's slope is −0.06221 kilograms a day on every day, so the left side is this.

```
7716.18 times 0.06221 = 480.02
```

About minus 480. Now the right side needs the weight at day 365.

```
0.06221 times 365 = 22.71
100 minus 22.71 = 77.29
```

His weight on the line is 77.29 kilograms, 22.71 below the start.

```
20 times 22.71 = 454.2
480 minus 454.2 = 25.8
```

The right side is about minus 25.8. The left side is minus 480. The two sides disagree by more than 450 kilocalories a day. The straight line starts at the right weight and is not a solution. It solves the equation only when ε is 0, that is, when expenditure ignores weight.

Now the curve that works.

```
W(t) = 100 - 24 × (1 - e^(-t/385.809))
```

Each is a ratio of the equation's own numbers.

```
480 divided by 20 = 24
7716.18 divided by 20 = 385.809
```

This is section 4's W = L plus (W0 minus L) $e^{(-kt)}$, with L = 76, W0 − L = 24 and k = 1 divided by 385.809.

Check it the way any candidate is checked: the start, then both sides. At t = 0, $e^{(0)}$ is 1. So 1 − 1 is 0, and W is 100. The initial value holds.

For the rate, use section 4's rule: the derivative of $e^{(-kt)}$ is $-k \times e^{(-kt)}$. The derivative of W is then this.

```
dW/dt = -24 × (1 divided by 385.809) × e^(-t/385.809)
24 divided by 385.809 = 0.06221
```

In symbols the check is two lines. The left side uses the rule for $e^{(-kt)}$; the right side is algebra.

```
left: 7716.18 × dW/dt = -480 × e^(-t/385.809)
right: -480 - 20 × (-24) × (1 - e^(-t/385.809)) = -480 + 480 - 480 × e^(-t/385.809)
```

Both come to minus 480 times $e^{(-t/385.809)}$. The deficit starts at 480 kilocalories a day and shrinks towards nothing as the weight falls.

Now walk it without the formula, one day at a time. On day 0 the weight is 100, so the bracket is 0 and the rate is the static rule's.

```
480 divided by 7716.18 = 0.062207
100 minus 0.062207 = 99.937793
```

On day 1 the weight is 99.937793, which is 0.062207 below the start.

```
20 times 0.062207 = 1.24414
480 minus 1.24414 = 478.75586
478.75586 divided by 7716.18 = 0.062046
99.937793 minus 0.062046 = 99.875747
```

About 99.8757 on day 2. The formula gives 99.8759. The steps drift a little from the curve, because each one holds the rate fixed for a whole day while the true rate keeps falling. A smaller step drifts less.

So a different step gives a different answer. By day 90, one-day steps give 95.00 and thirty-day steps 94.83; the formula gives 95.01. A model run is also a choice of step. The next rung shows when too large a step makes the answer run away.

Run the formula for three years and put the static line beside it. A second curve uses the top of Hall and Guo's range, ε = 30.

```
day    static rule, ε = 0 ε = 20 kcal/day per kg ε = 30 kcal/day per kg
0    100    100    100
90    94.40    95.01    95.28
180    88.80    91.05    91.95
365    77.29    85.32    87.87
545    66.10    81.84    85.92
730    54.59    79.62    84.94
1095 31.88    77.40    84.23
```

All three columns come from one equation. Only ε differs. With ε at 0, the loss never slows. With any ε above 0, the loss slows and flattens.

The same equation can also be run backwards. Polidori and colleagues (2016) had weights and no reliable intake. They put the weight and its rate into their Equation 1 and read off ΔEI. The intake change they report is an output of the equation, not a measurement.

Figures quoted in this illustration are from [32, 25, 57, 10].

WHERE THIS PICTURE BREAKS. The value of ρ here is the static rule's 7,716.18 kilocalories per kilogram, borrowed as a stand-in. For a real body, ρ "need not be a constant", as Chow and Hall

(2008) put it. It changes with what the weight lost is made of. So the curves in the table have the right shape and not Hall's timing.

The straight line for expenditure is an approximation. It is safest for changes small enough that one slope describes them.

MUST KNOW POINTS FOR YOU.

- The energy balance identity predicts nothing on its own. A prediction appears only when someone writes expenditure, and perhaps intake, as a rule in weight. When you read a projection built "on energy balance", find that rule.
- A curve that starts at the right weight is not thereby a solution. Check a claimed solution at the start and at a later time, on both sides of the equation.
- "Cut 480 kilocalories and you run a 480 deficit every day" is the static rule. Correct a trainee who carries the day-one deficit forward: the cut stays the same; the deficit does not.
- When you step an equation forward, work the rate out again from the new state at every step.
- Keep the units in step: ρ in kilocalories per kilogram and intake in kilocalories a day give kilograms a day.
- Use it for the shape of a response and for modest changes. For one person's timetable over years, use a validated model, and say in writing which one you used.
- When a paper reports intake "calculated" from repeated weighings, read it as the output of an equation. It moves with the ρ and the ε put in.

EXERCISE 1 (interpretation). Polidori and colleagues (2016) calculate a change in energy intake from repeated weighings with their Equation 1. The paper writes BW for body weight. A letter or digit after a symbol is a label: i marks the interval, and 0 the start.

```
ΔEIi = ρ × (dBWi/dt) + ε × (BWi - BW0) + (Δδ divided by (1 - β)) × BW0 + UGE
```

For each symbol, say what it stands for and its unit, as far as the paper tells you. Then say which two terms this section set to zero, and what setting each to zero assumes.

EXERCISE 2 (teaching). A first-year resident has ten minutes and a whiteboard. They ask: "If energy in minus energy out is always true, why does the same diet give smaller losses each month?" Explain, with the equation, and expect them to want the mechanism.

PRACTICE. Sixteen problems on this technique, easiest first. Work them on paper before turning to the answers, which are in the appendix at the back under this section's name.

1. A rule gives the rate of change of y as dy/dt = 6 − 0.5 × y. Work out the rate when y is 4, when y is 12, and when y is 20.

2. Say which of these are differential equations. For each one that is, say whether the rate depends on the state y, or on the time t alone.

1. $y = 3t + 2$
2. $dy/dt = 3t + 2$
3. $dy/dt = -0.2 \times y$
4. $E = 4P + 9F + 4C$

3. Take $dy/dt = 6 - 0.5 \times y$ with $y = 4$ at $t = 0$. Take three forward steps of size 1. Give y at t = 1, 2 and 3.

4. Take the initial-value problem $dy/dt = 6 - 0.5 \times y$, with $y = 4$ at $t = 0$. Check whether $y = 12 - 8 \times e^{(-0.5t)}$ solves it. Then compare its value at t = 2 with the forward steps you would get from step size 1.

5. Three people offer a solution to $dy/dt = 6 - 0.5 \times y$ with $y = 20$ at $t = 0$.

1. $y = 12 + 8 \times e^{(-0.5t)}$
2. $y = 20 - 4t$
3. $y = 12 + 4 \times e^{(-0.5t)}$

Which one solves the initial-value problem? Say what is wrong with each of the other two.

6. Hall and Guo (2017) use a cut of 300 kilocalories a day from the diet. On the first day of the cut, the weight has not yet moved, so the whole cut is the deficit. With $\rho = 7{,}716.18$ kilocalories per kilogram, the static rule's figure, work out dW/dt in kilograms a day and in kilograms a week.

7. Same cut of 300 kilocalories a day, same $\rho = 7{,}716.18$ kilocalories per kilogram. Hall and Guo put the slope of expenditure against weight at 20 to 30 kilocalories a day per kilogram. A person, made up for this problem, is now 4 kilograms below their starting weight. Work out dW/dt at each end of that range.

8. Polidori and colleagues computed dW/dt from weighings 52 days apart. A person, made up for this problem, weighed 98.0 kilograms at the start. They weighed 96.4 kilograms at one weighing and 95.8 kilograms at the next, 52 days later.

Use $\rho = 7{,}716.18$ kilocalories per kilogram and $\varepsilon = 30$ kilocalories a day per kilogram, Polidori's round figure for the change in expenditure. Take $W - W0$ at the later weighing. Work out ΔEI with $\rho \times dW/dt = \Delta EI - \varepsilon \times (W - W0)$.

9. Polidori and colleagues describe a commercial weight-loss programme. "After 1 year, the average energy intake was practically at baseline levels while body weight was still reduced by ~5 kg."

Take $\Delta EI = 0$ and $W - W0 = -5$ kilograms (the ~ means about). Use $\rho = 7{,}716.18$ kilocalories per kilogram and ε at each end of Hall and Guo's 20 to 30 kilocalories a day per kilogram. Work out dW/dt, and the change over the next 30 days if it held. Say which way the weight is heading.

10. Hall and colleagues' 100 kg man cuts intake by 480 kilocalories a day. Use ρ = 7,716.18 kilocalories per kilogram and ε = 20 kilocalories a day per kilogram. Take three forward steps of 30 days each, and give his weight at day 90. Then give the static rule's weight at day 90, which is 100 − 480 × 90 / 7,716.18.

11. Here is a worked answer. Find the step that broke.

```
the man's equation is 7716.18 × dW/dt = -480 - 20 × (W - 100), starting at 100 kg
on day 0, W - 100 is 0, so dW/dt = -480 / 7716.18 = -0.062207 kg a day
over 90 days: 0.062207 × 90 = 5.5986 kg
so the equation predicts 100 - 5.5986 = 94.40 kg at day 90
```

12. Here is a worked answer. Find the step that broke.

```
a person cuts intake by 300 kcal a day; ρ = 7716.18 kcal/kg; ε = 20
model: ρ × dW/dt = ΔEI + ε × (W - W0)
4 kg below the start: -300 + 20 × (-4) = -380
dW/dt = -380 / 7716.18 = -0.04925 kg a day
so at 4 kg down the loss has sped up from 0.0389 to 0.0492 kg a day
```

13. Here is a worked answer. Find the step that broke.

```
Hall's man cuts intake by 2 MJ a day
ρ = 7716.18 kcal per kg
day one: dW/dt = -2 / 7716.18 = -0.000259 kg a day
over a year: 0.000259 × 365 = 0.0946 kg
so the cut barely moves his weight
```

14. Here is a worked answer to problem 8. The person weighed 98.0 kilograms at the start, then 96.4 and 95.8 kilograms at weighings 52 days apart. Take ρ = 7,716.18 and ε = 30. Find the step that broke.

```
dW/dt = (95.8 - 96.4) / 52 = -0.011538 kg a day
W - W0 = 95.8 - 96.4 = -0.6 kg
ΔEI = 7716.18 × (-0.011538) + 30 × (-0.6) = -89.03 - 18 = -107.03 kcal a day
```

15. A commentator writes: "Energy balance is only an accounting identity. Any weight prediction built on it is circular, because it just restates that calories in minus calories out equals the change in stores." The commentator is made up for this problem.

Decide what to compute, compute it, and say what your answer does not establish.

16. A review sentence reads: "Polidori and colleagues measured how much people's eating changed, using weights taken 52 days apart." The review is made up for this problem; the paper is real.

Decide what to compute, compute it, and say what your answer does not establish.

8 · Equilibrium: where the rate is zero

DEFINITION. An equilibrium, or steady state, of a differential equation dW/dt = f(W) is a constant value of the state at which the rule gives a rate of zero. It is written W, *read "W star", so f(W) = 0*. To find the equilibria, set the right-hand side equal to zero and solve for the state.

Take the equation of the last section, $\rho \times dW/dt = \Delta EI - \varepsilon \times (W - W0)$. Setting its right side to zero gives $\varepsilon \times (W^* - W0) = \Delta EI$. So $W^* - W0 = \Delta EI \div \varepsilon$.

When intake also depends on weight, it rises by a fixed number of kilocalories a day for each kilogram lost. Polidori and colleagues write that number kP, k with a P for proportional. They measured it in people with type 2 diabetes taking a drug that makes them lose glucose in urine. The model is then $\rho \times dW/dt = \Delta EI - (\varepsilon + kP) \times (W - W0)$. Here ΔEI is the shift the diet makes; the change appetite makes is the kP term. The two slopes add, and the shift becomes $\Delta EI \div (\varepsilon + kP)$. Hall and Guo call a model with feedback on both intake and expenditure a set point model. The name is theirs; the equilibrium still moves when intake is shifted.

In either model, a weight that starts away from the equilibrium approaches it along an exponential. Its time constant, written tau (τ), is ρ divided by the total slope, in days. The gap shrinks as $e^{(-t/\tau)}$, so τ is 1 divided by k, the rate constant of section 4. The gap halves every $\tau \times \ln 2$ days.

An equilibrium is stable if a state pushed a little away from it returns to it, and unstable if the state moves further away. [58, 59, 60, 61, 62, 63, 64, 63, 65, 66, 67, 68]

IN PLAIN TERMS. A weight stops moving when its rate of change is zero. So take the equation, put zero where the rate is, and solve for the weight. That weight is the equilibrium.

Picture it as two lines on one graph, with weight along the bottom and kilocalories a day up the side.

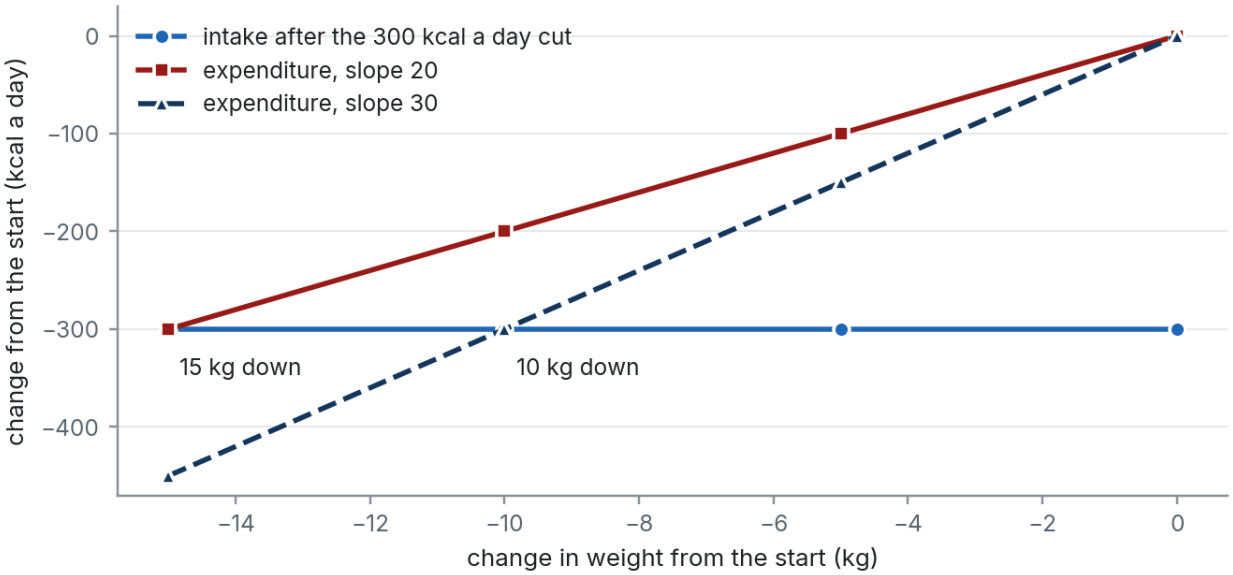
- Expenditure rises as weight rises.
- Intake is a flat line, if it ignores weight.

Where the two lines cross, intake equals expenditure, and the weight stops moving. Cut intake, and the flat line drops. The crossing slides to a lower weight. How far it slides is the cut divided by the slope of the expenditure line.

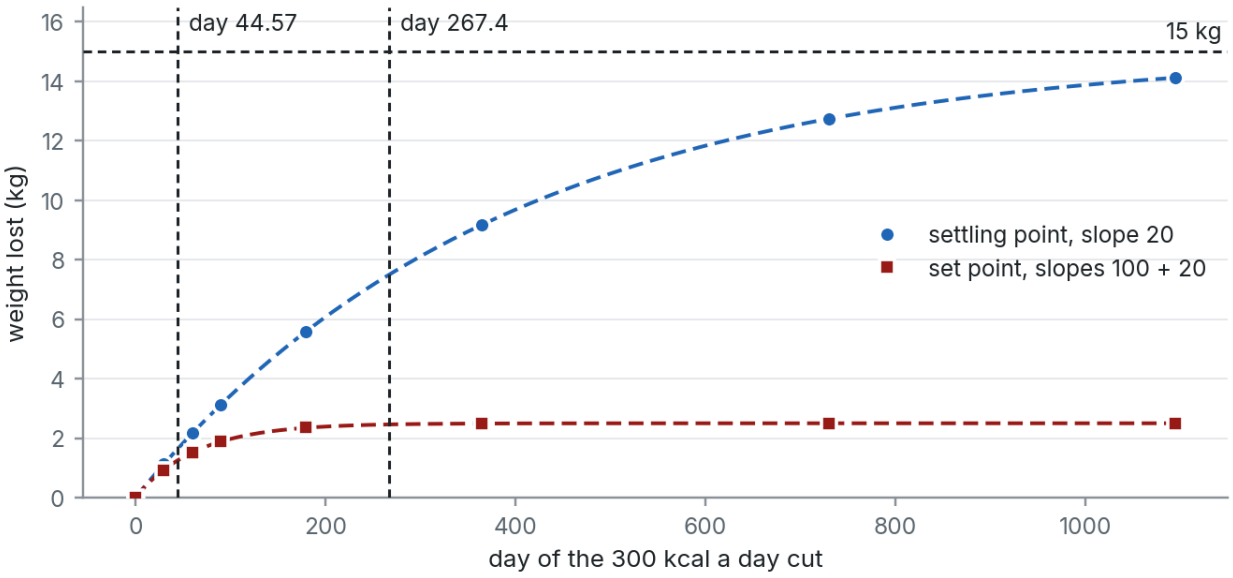
Now let intake rise as weight falls, which is what appetite does. The two slopes add, and the crossing slides a much shorter way.

- The weight does not arrive at the crossing on a date. It closes the gap by the same fraction every day, fast at first and then ever more slowly.

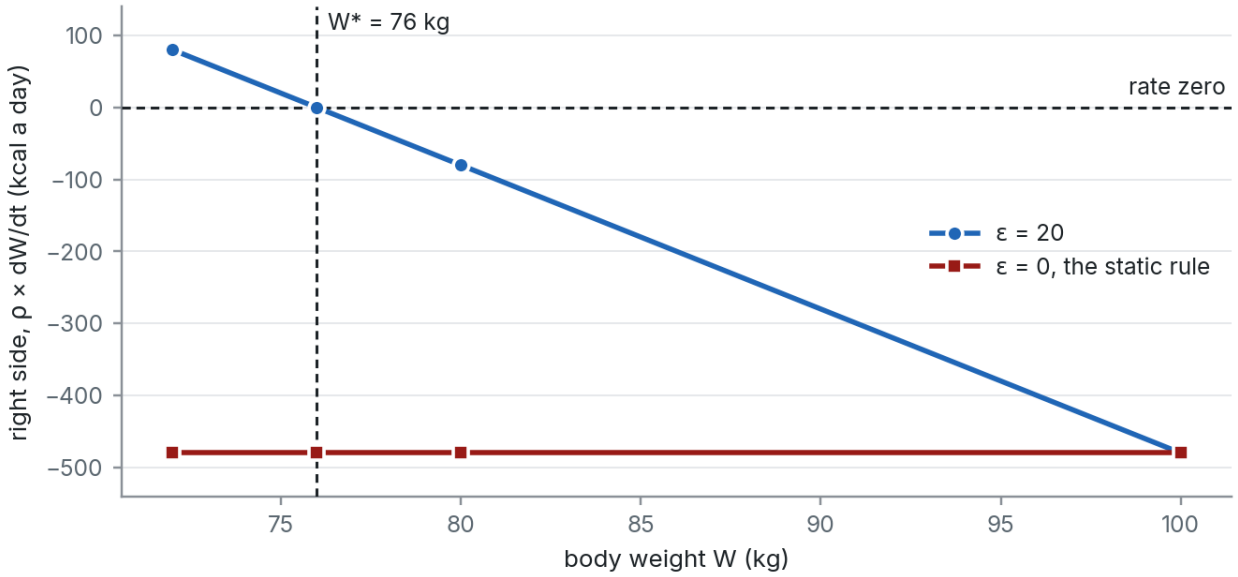
- Push the weight a little either side of the crossing, and the rate sends it back. That is what stable means here.
- At the crossing, intake and expenditure are equal. Both are still large.



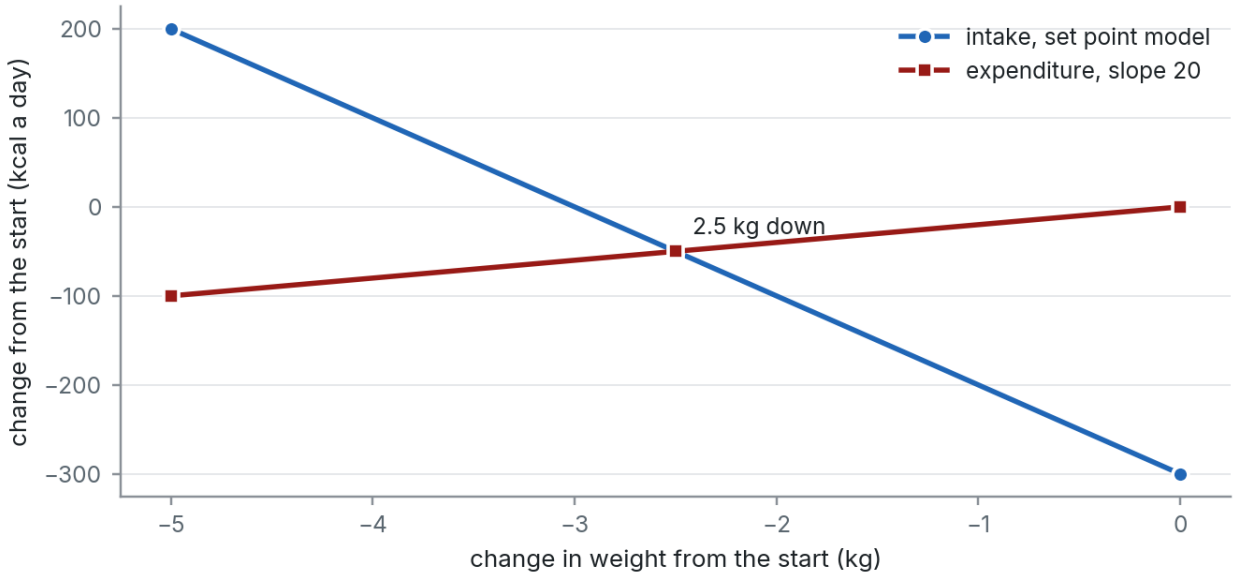
Hall and Guo's settling point model after a 300 kcal a day cut. Flat intake crosses expenditure 15 kg down when the slope is 20 kcal a day per kg, and 10 kg down when it is 30.



Loss after the same 300 kcal a day cut, with ρ = 7,716.18: the settling point model (slope 20) is still closing on 15 kg at day 1095, and reaches half of it only at day 267.4. With intake also responding (slopes 100 + 20) the loss is halfway to 2.5 kg by day 44.57 and flat within months.



The right side of Hall's man's equation after a 480 kcal a day cut. With ε = 20 it is 80 at 72 kg, 0 at 76 kg and −80 at 80 kg: positive below the equilibrium and negative above it, so the weight is pushed back to 76 from either side. With ε = 0 it is −480 at every weight and never reaches zero.



The set point model after a 300 kcal a day cut. Intake now rises by 100 kcal a day for each kilogram lost, so it meets expenditure (slope 20) only 2.5 kg down, against 15 kg when intake is flat.

ILLUSTRATION. Start with the model that has no sensible equilibrium: the static rule. Its expenditure ignores weight, so ε is 0. Hall and colleagues' man cuts 480 kilocalories a day. Set the rate to zero in his equation and see what is left.

```
0 = -480 - 0 × (W* - 100)
0 = -480
```

That is false for every weight. There is no equilibrium, so the weight falls for ever, which is the static rule's straight line. Now let him go back to his old intake, so ΔEI is 0.

```
0 = 0 - 0 × (W* - 100)
```

This is true for every weight. Every weight is an equilibrium, so whatever he weighs, he stays there.

Now use the equation of the last section, with ε = 20 kilocalories a day per kilogram.

```
0 = -480 - 20 × (W* - 100)
20 × (W* - 100) = -480
W* - 100 = -480 ÷ 20 = -24
W* = 76
```

The equilibrium is 76 kilograms.

You can check that 76 is stable. Put the man 4 kilograms above it, at 80, and 4 below it, at 72.

```
at 80: -480 - 20 × (80 - 100) = -480 + 400 = -80
at 72: -480 - 20 × (72 - 100) = -480 + 560 = 80
```

Above 76 the rate is negative, so the weight falls back. Below 76 it is positive, so the weight rises back.

Now the two models Hall and Guo (2017) draw side by side.

In the settling point model, intake is flat and expenditure rises “with a slope of about 20–30 kcal/d per kg”. The shift in the equilibrium is the cut divided by the slope.

```
300 divided by 20 = 15
300 divided by 30 = 10
```

Ten to 15 kilograms lower.

In the set point model, intake also depends on weight, “with a slope of about -100 kcal/d per kg”. Write the model with both slopes.

```
ρ × dW/dt = -300 - 100 × (W - W0) - ε × (W - W0)
= -300 - (100 + ε) × (W - W0)
```

Hall and Guo’s −100 is intake against weight, so a fall in weight raises intake by 100 a kilogram. In intake minus expenditure it enters as −100 × (W − W0), and expenditure’s +20 enters as −20 × (W − W0). The two terms have the same sign, so their sizes add. Set the rate to zero, with ε = 20 and then 30.

```
0 = -300 - 120 × (W* - W0)
120 × (W* - W0) = -300
W* - W0 = -300 ÷ 120 = -2.5
W* - W0 = -300 ÷ 130 = -2.307692308
```

About 2.3 to 2.5 kilograms, against 10 to 15.

The last section’s solution can be written as a gap that shrinks.

```
W(t) - W* = (W0 - W*) × e^(-t/τ),    with τ = ρ ÷ (total slope)
```

Take ρ = 7,716.18 kilocalories per kilogram, the stand-in of the last section. For the settling point model with slope 20 the total slope is 20. For the set point model it is 100 + 20 = 120.

```
7716.18 divided by 20 = 385.809
7716.18 divided by 120 = 64.3015
```

Time constants of about 386 and 64 days. Each gap halves every τ × ln 2 days. From the section on e, ln 2 is 0.6931.

```
385.809 times 0.6931 = 267.4
64.3015 times 0.6931 = 44.57
```

Here is the loss, in kilograms, after Hall and Guo's 300 kilocalorie cut, with each model.

day	settling point, slope 20	set point, slopes 100 + 20
0	0	0
30	1.12	0.93
60	2.16	1.52
90	3.12	1.88
180	5.59	2.35
365	9.18	2.49
730	12.74	2.50
1095	14.12	2.50

The set point curve flattens sooner and lower.

And if intake goes back to where it was, the equilibrium goes back to the starting weight. The weight follows it, along the same kind of curve. Regain here is the equation working, not a person failing.

Figures quoted in this illustration are from [32, 25, 10, 57].

WHERE THIS PICTURE BREAKS. Every equilibrium here comes from straight lines. The slopes are group averages, and Hall and Guo expect "a wide degree of individual variation".

The time constants use the static rule's ρ as a stand-in, so their sizes are illustrative.

A scale does not show an equilibrium over the short term. Hall and colleagues (2012) warn that "changes in body weight also include changes in body water, which may be variable".

MUST KNOW POINTS FOR YOU.

- To find where a model says weight will settle, set the rate to zero and solve for the weight. Do not run the model forward and read off where it seems to stop.

- In the settling point model, a lasting cut of 300 kilocalories a day moves the equilibrium down 10 to 15 kilograms, by Hall and Guo's slopes. Before you give a patient a target from a cut in food, say which model it came from.
- "Go back to your old intake and you will stay at your new weight" is the static model. When expenditure falls with weight, going back to the old intake moves the equilibrium back to the old weight. Tell a patient that regain is this equation at work.
- A steady weight means intake equals expenditure, not that either is small.
- A weight that stops falling for a few weeks is not thereby at an equilibrium. The rate is zero at an equilibrium and stays zero. It can pass through zero for a moment while intake is still moving, and water shifts can hide the true rate over the short term.
- An equilibrium worked out from straight lines is only as good as the lines. So quote such a figure as the model's answer for a group, and name the slopes.
- When a trainee adds up the slopes, check the signs first. Both push the weight back, so their slopes add. Subtract them, and the pull looks weaker and the shift looks larger.

EXERCISE 1 (critique). A policy note says this.

> *A population-wide cut of 100 kcal a day, sustained, will lower average weight every year. The loss is 100 × 365 ÷ 7,700 = 4.7 kg a year.*

Say what is wrong, using the idea of an equilibrium. Then give the figure you would put in its place, with the source of every number in it.

EXERCISE 2 (teaching). A journalist has three minutes and no jargon. They ask: "Why do people who lose weight so often put it back on?" Give them the idea of an equilibrium without the word, and one quotable sentence.

PRACTICE. Seventeen problems on this technique, easiest first. Work them on paper before turning to the answers, which are in the appendix at the back under this section's name.

1. Find the equilibrium of dy/dt = 12 − 0.4 × y.

2. Find the equilibrium of dy/dt = −0.05 × (y − 70). Then work out the rate at y = 60 and at y = 80, and say what each tells you.

3. Intake is a flat line at 2,000 units. Expenditure is 500 + 25 × W. Find the W where they cross. Then drop intake to 1,800 and find the new crossing. How far did it move?

4. Intake is 5,700 − 60 × W and expenditure is 500 + 20 × W. Find the crossing. Then lower intake by 400 at every W and find the new crossing.

5. Take dy/dt = −0.02 × (y − 50), with y = 60 at t = 0. Give the time constant, the half-time, and y at t = 50. Use ln 2 = 0.6931 and $e^{(-1)}$ = 0.3679.

6. A flat intake line is lowered by 300. The equilibrium moves by 12. What is the slope of the expenditure line?

7. Hall and Guo (2017) cut 300 kilocalories a day from the diet in their settling point model. Expenditure rises with weight at about 20 to 30 kilocalories a day per kilogram. Work out how far the equilibrium weight moves at each end of that range.

8. Now Hall and Guo's set point model. The same 300 kilocalories a day is cut, and intake also changes with weight, with a slope of about −100 kilocalories a day per kilogram. Work out the shift in the equilibrium at each end of the 20 to 30 range for expenditure. Compare with the settling point model.

9. Hall and colleagues (2011) give a rule of thumb. Every change of intake of 100 kJ a day leads to "an eventual bodyweight change of about 1 kg". The paper adds "(equivalently, 10 kcal per day per pound of weight change)". Turn each form into kilocalories a day per kilogram. Then give the equilibrium shift each form predicts for a cut of 300 kilocalories a day. One kilocalorie is 4.184 kilojoules. One pound is 0.45359237 kilograms.

10. Polidori and colleagues ran a drug trial. At steady state, "energy intake was calculated to have increased by ~350 kcal/day" (about 350). Their Equation 4 says the change in intake that appetite makes, which they write ΔEI, is minus kP times the change in weight. kP is about 100 kilocalories a day per kilogram. Work out the weight change at steady state. Compare it with the paper's "new equilibrium several kilograms lower".

11. Hall and colleagues' 100 kg man cuts 480 kilocalories a day. Their model predicts a plateau "at about 75 kg". Their rule of thumb, 100 kilojoules a day per kilogram, is 23.90 kilocalories a day per kilogram. Work out the plateau the rule of thumb gives him. Then say why the two might differ. The paper's summary helps: "adults with greater adiposity have a larger expected weight loss for the same change of energy intake". Adiposity means how much body fat a person carries.

12. Here is a worked answer. Find the step that broke.

```
set point model: 300 kcal/day cut, intake slope -100, expenditure slope 20
combined slope = 100 - 20 = 80
shift = 300 ÷ 80 = 3.75 kg
```

13. Here is a worked answer. Find the step that broke.

```
settling point model: ρ = 7716.18 kcal per kg, ε = 20 kcal/day per kg
time constant τ = ε ÷ ρ = 20 ÷ 7716.18 = 0.00259
so the weight is essentially at its new equilibrium within a day
```

14. Here is a worked answer. Find the step that broke.

```
Polidori's programme: weight lowest at about 8 months, with intake within 100 kcal/
day of baseline
at the lowest point, weight stopped falling, so dW/dt = 0
dW/dt = 0 is the definition of an equilibrium
so the lowest weight was the new equilibrium, and the group could hold it on that
intake
```

15. Here is a worked answer. Find the step that broke.

```
Hall's man, ε = 20, ρ = 7716.18: equilibrium 76 kg, time constant 385.809 days
the time constant is how long it takes to reach equilibrium
so on day 386 he weighs 76 kg
```

16. A newspaper headline reads: "Cut 300 calories a day and lose 15 kg, says top weight scientist." The headline is made up for this problem.

Decide what to compute, compute it, and say what your answer does not establish.

17. A colleague says of a patient: "Her weight has been flat at 82 kg for two months. She's hit her set point, so no change in what she eats will move it." The patient and her weight are made up for this problem.

Decide what to compute, compute it, and say what your answer does not establish.

9 · Highest and lowest points: slope zero and curvature

DEFINITION. A critical point of f is an input c inside the domain, not at one of its ends, at which the derivative f′(c) is zero or does not exist.

The first derivative test sorts a critical point c by the sign of f′ on either side. If f′ changes from positive to negative as x rises through c, f has a local maximum at c. If it changes from negative to positive, f has a local minimum. If it keeps the same sign, neither.

Suppose f′(c) = 0, and f″ is continuous near c. If the second derivative f″(c) is greater than zero, f has a local minimum at c. If f″(c) is less than zero, f has a local maximum. If f″(c) = 0, this test decides nothing, and the first derivative test has to be used instead. This is the second derivative test. [69, 70, 71, 72, 73, 74, 75, 76, 77, 78]

IN PLAIN TERMS. Picture a smooth hill. At the very top, the ground is level for an instant. The same is true at the bottom of a smooth valley.

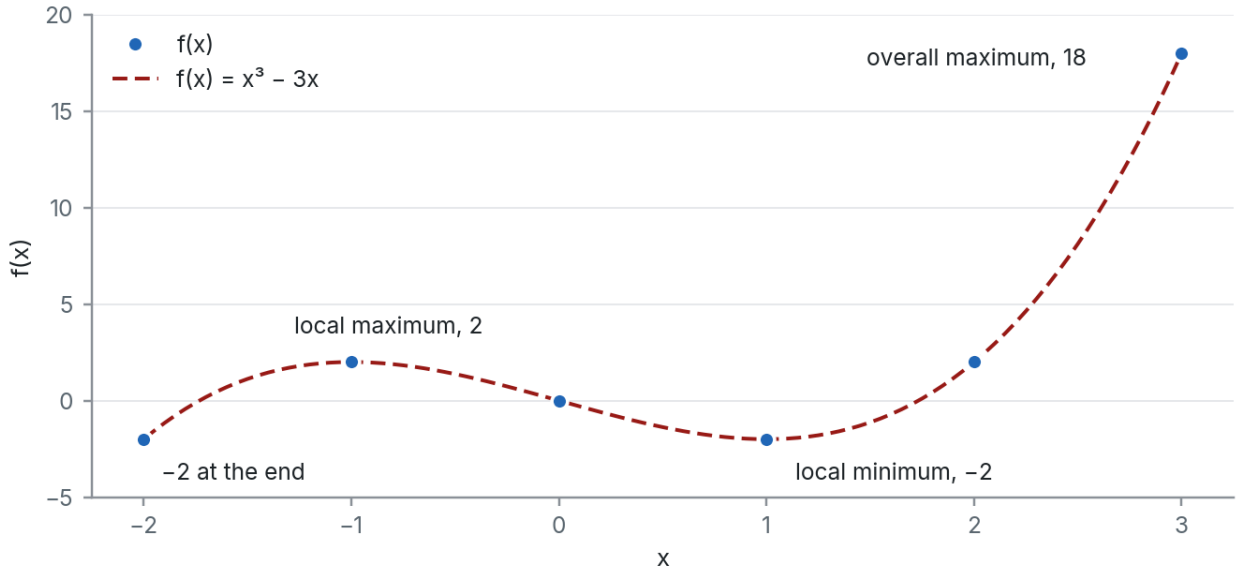
So to find a top or a bottom, look for where the slope is zero. In the words of section 2, look for where the derivative is zero.

Level ground does not tell you which of the two you are standing on. Walk across a hilltop and the slope goes from uphill to downhill, so the slope is falling. The second derivative from section 3 is the rate at which the slope changes. Positive means a valley floor. Negative means a hilltop.

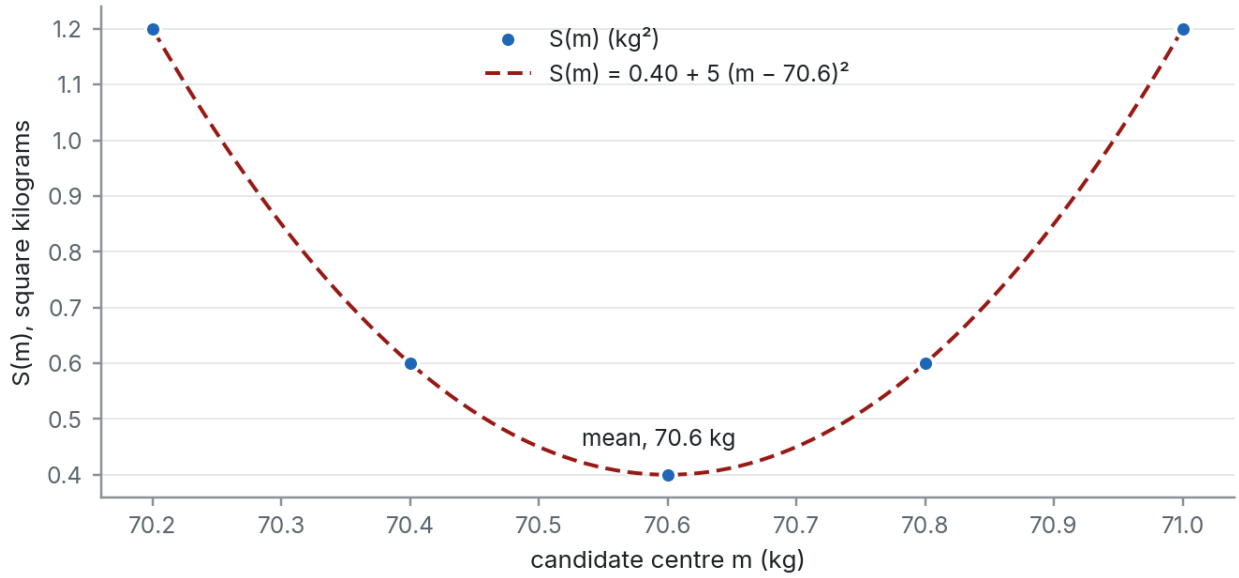
Ground can be level without being a top or a bottom. A flat landing on a staircase is level, and the stairs keep climbing after it. And suppose you may walk only part of the hill. Then the highest place you can reach may be the edge of that part, where the ground is still sloping upward.

A top that is higher than the ground near it is a local maximum. The highest point over the whole stretch is the overall maximum.

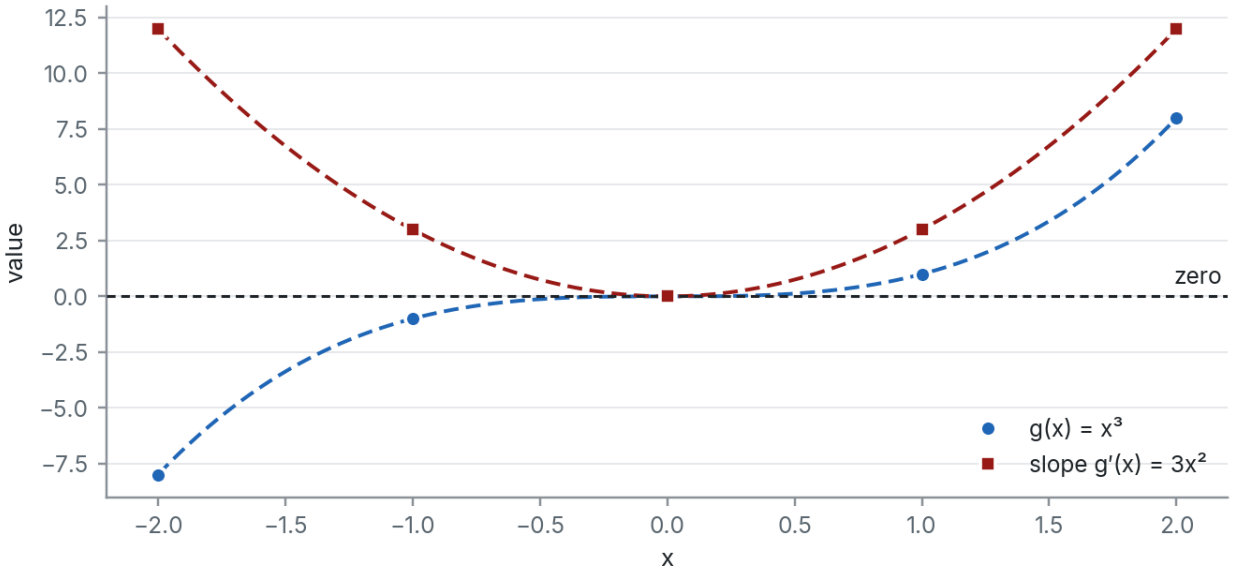
You have several readings of one thing and want a single number to stand for them. Choose the number that makes the misses, each squared and then added up, as small as they can be. That number turns out to be the mean.



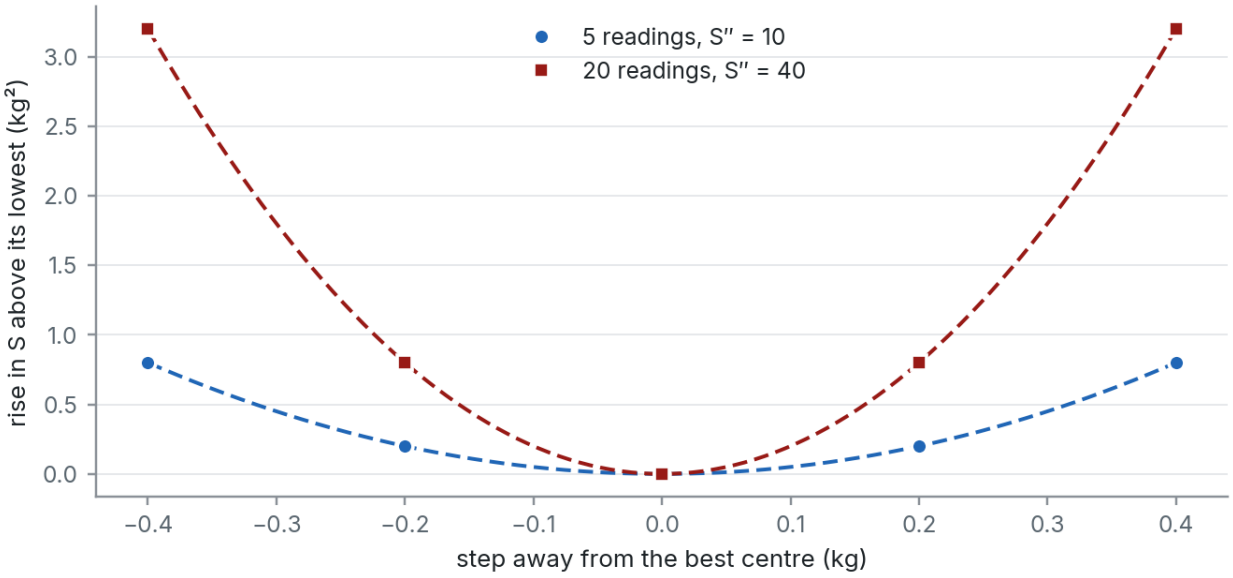
f(x) = x³ − 3x for x from −2 to 3. The slope is zero at x = −1 and at x = 1, but the highest value over the interval, 18, is at the end x = 3.



The five made-up weighings. For each candidate centre m, S(m) is the sum of squared misses. It is lowest, 0.40 square kilograms, at the mean of 70.6 kg, and rises by 5 times the square of the step on either side.



The plain cube $g(x) = x^3$ and its slope $g'(x) = 3x^2$. The slope is zero at $x = 0$ and positive on both sides of it, so g rises before 0 and after it. Level ground, and neither a top nor a bottom.



How far S climbs above its lowest value as the centre steps away from the best one. With the five weighings S″ is 10, so a step of 0.4 kg costs 5 times 0.16, which is 0.80 kg². With 20 readings S″ would be 40, and the same step would cost 20 times 0.16, which is 3.2 kg².

ILLUSTRATION. Start with the way this goes wrong. Take $f(x) = x^3 - 3x$, and suppose you want its highest and lowest values for x anywhere from −2 to 3. Differentiate with the power, constant-multiple and sum rules of section 3.

```
f′(x) = 3x² - 3
```

Set the derivative to zero and solve.

```
3x² - 3 = 0
3x² = 3
x² = 1
1 times 1 = 1
-1 times -1 = 1
x = 1 or x = -1
```

The second derivative sorts them.

```
f″(x) = 6x
f″(-1) = -6, less than zero, so a local maximum
f″(1) = 6, greater than zero, so a local minimum
```

It is tempting to stop here and write “the maximum is at x = −1”. Do not. Work out f at both critical points and at both ends of the interval.

x	f(x)
-2	-2
-1	2
0	0
1	-2
2	2
3	18

The overall maximum over this interval is 18, at x = 3. That is an end of the interval, and the slope there is nowhere near zero. The overall minimum is −2. It is reached twice: at the local minimum x = 1, and at the end x = −2.

The local maximum at x = −1 is only a peak among its neighbours, with a value of 2.

Some problems tie two unknowns by a fixed amount, such as a total length of fence. Write one unknown in terms of the other before you differentiate.

One more trap sits at x = 0 for the plain cube, $g(x) = x^3$. There $g'(x) = 3x^2$ is zero, and $g''(x) = 6x$ is zero too, so the second derivative test decides nothing. Go back to the sign of g′ on each side. $3x^2$ is positive for every x except 0. So g rises before 0 and rises after it.

A person is weighed on five mornings of one week, on the same scale. The readings are made up for this example: 70.4, 70.6, 70.2, 70.8 and 71.0 kg.

One answer: choose the number m that makes the misses, squared and added up, as small as they can be. Call that total S(m), said “S of m”.

```
S(m) = Σ[i = 1 to n] (xi - m)²
```

The capital sigma, Σ, from section 1, says: add this up for i from 1 to n, over the n readings x1 to xn. From here on the counter is left off: Σ x means Σ[i = 1 to n] xi.

S(70.4) is 0.60 square kilograms, written kg^2. The unit is squared because each miss, in kilograms, was multiplied by itself.

m (kg)	S(m) (kg^2)
70.2	1.20
70.4	0.60
70.6	0.40
70.8	0.60
71.0	1.20

The smallest S in the table is at 70.6. A table only tests the candidates in it. The derivative shows that no m anywhere does better, for any set of n readings.

Open the square first.

```
(x - m)² = (x - m) times (x - m)
(x - m)² = x times x - x times m - m times x + m times m
(x - m)² = x² - 2mx + m²
```

Now add that up over all n readings. The candidate m is the same number in every term, so it comes outside the sum. And m^2 is added once for each reading, n times in all. On the five weighings, the two moves look like this.

```
-2m times 70.4 - 2m times 70.6 - … - 2m times 71.0 = -2m times (70.4 + 70.6 + … +
71.0)
m² + m² + m² + m² + m² = 5m²
```

For any n readings, this gives the following.

```
S(m) = Σ x² - 2m Σ x + n m²
```

Read the formula term by term: the number in front of m^2 is n, and the number in front of m is $-2 \Sigma x$.

Differentiate with respect to m, using the rules of section 3. The first term, Σx^2, has no m in it, so its derivative is zero. The second is the fixed number $-2 \Sigma x$ times m, so its derivative is $-2 \Sigma x$. The third is n times m^2, so its derivative is 2n times m.

```
S′(m) = -2 Σ x + 2nm
```

Set it to zero and solve for m.

```
2nm = 2 Σ x
m = Σ x divided by n
```

Σx divided by n is the sum of the readings divided by how many there are. That is the mean of Book 0, D5. Differentiate once more to see which kind of critical point it is.

```
S″(m) = 2n
```

2n is positive for any count of readings, so the mean is a minimum. It is the only critical point, so it is the overall minimum as well.

Choosing a number by making squared misses as small as possible is called least squares. The mean is the least-squares answer when the thing you choose is a single number.

Next, the size of the second derivative. Take any quadratic, $f(x) = px^2 + qx + r$, with its level point at c. Step a distance h away from c and expand.

```
f(c + h) = p(c + h)² + q(c + h) + r
f(c + h) = pc² + 2pch + ph² + qc + qh + r
f(c + h) = f(c) + (2pc + q)h + ph²
```

The bracket 2pc + q is f′(c), which is zero at a level point, so the middle term drops out. And f″ is 2p, so p is half of f″.

```
f(c + h) = f(c) + half of f″(c) times h²
```

For the weighings, S″ is 2n, which is 10, so half of it is 5. S rises from its lowest value of 0.40 by 5 times the square of the step.

A step of 0.4 kg, to 70.2 or 71.0, raises S by 5 times 0.16, which is 0.80, as the table shows.

```
0.4 times 0.4 = 0.16
5 times 0.16 = 0.80
0.40 plus 0.80 = 1.20
```

With 20 readings instead of 5, S″ would be 40. The same step would then raise S by 20 times 0.16, not 5 times 0.16.

The more readings, the sharper the valley. A sharp valley pins the best value down: a small step off it costs a lot.

Last, the lowest value of S itself, 0.40 kg^2. That is the sum of squared deviations from the mean. Book 0, D5 divided it by one less than the count to get the sample variance.

```
0.40 divided by 4 = 0.1
```

The sample variance of the five weighings is 0.1 kg^2, and their standard deviation is its square root, about 0.32 kg.

Keep all this apart from the equilibrium of section 8. Suppose a person’s body weight rose for some weeks after a change in their life, then turned and fell. At the top of that curve, the rate of change of body weight with respect to time was zero for an instant. The weight did not stay there. A zero slope for an instant marks a turning point. A rate held at zero marks an equilibrium.

WHERE THIS PICTURE BREAKS. The hill picture needs a smooth curve. At a sharp corner, a top can have no slope at all. That is the other kind of critical point, where the derivative does not exist, and setting a derivative to zero will never find it.

The rule that a step h raises f by half of f″(c) times h^2 is exact only for a quadratic. For other curves it is close for small steps and drifts for larger ones.

The mean wins only because the misses were squared before they were added. Squaring makes one big miss count for a great deal. Add up the sizes of the misses without squaring them, and the median of Book 0, D5 can come out best.

MUST KNOW POINTS FOR YOU.

- A derivative of zero marks a candidate, not an answer. It can be a minimum, or a level step with neither. Check the second derivative, or the sign of f′ on each side, before you name what you found.
- Over an interval of inputs, work out the value at both ends as well as at every critical point. When a study reports that the best dose was the highest dose it tested, that is an end of the tested doses. It is not a peak, and higher doses remain untested.
- The derivative test finds local answers: a point lower than its near neighbours. Compare every candidate before you call one the overall minimum.
- Because the misses are squared, one far-off reading pulls the answer toward itself. Look at the readings before you accept a least-squares figure.
- When a paper reports a best-fit value, ask how fast the fit gets worse away from it. A flat valley means the data barely pin the value down.
- A slope of zero on a weight curve is not an equilibrium. Call a weight an equilibrium only when the rule that gives the rate stays at zero there.

EXERCISE 1 (interpretation). Below is the derivation from this section that the mean makes the sum of squared deviations smallest. The move taken at each line has been left out, and one line has been removed.

```
S(m) = Σ[i = 1 to n] (xi - m)²
S(m) = Σ x² - 2m Σ x + n m²
S′(m) = -2 Σ x + 2nm
m = Σ x divided by n
S″(m) = 2n
```

Write beside each line the one move that produced it from the line before. Then write the missing line in its place, with its move.

PRACTICE. Sixteen problems on this technique, easiest first. Work them on paper before turning to the answers, which are in the appendix at the back under this section's name.

1. $f(x) = x^2 - 6x + 11$. Find the input at which f′(x) is zero. Say whether f has a local maximum or a local minimum there, and give the value of f at that point.

2. $g(x) = -2x^2 + 8x + 1$. Find the critical point, say which kind it is, and give its value.

3. $h(x) = x^3 - 12x$. Find both critical points, sort each with the second derivative, and give the value of h at each.

4. $f(x) = x^4$. Show that $x = 0$ is a critical point and that the second derivative test decides nothing there. Then decide what kind of point it is.

5. $f(x) = x^2 - 4x + 5$, for x from 0 to 5. Find the overall maximum and the overall minimum over that interval, and where each occurs.

6. The five values are 3, 7, 8, 12 and 5. Find the number m that makes $\Sigma\,(x - m)^2$ smallest. Give the smallest value of that sum, and the second derivative of the sum with respect to m.

7. A nurse measures one patient's waist three times in a row. The readings are made up for this problem: 88.4, 89.0 and 88.7 cm.

Find the single value that makes the sum of squared misses smallest, and that smallest sum, with its unit. Then find the sum of squared misses if 89.0 cm were reported instead.

8. A municipality has 400 metres of fencing for a rectangular walking ground in a park. One long side runs along an existing wall and needs no fence. The figures are made up for this problem.

Which side lengths give the largest area, and what is that area?

9. Two people are weighed repeatedly on the same scale. Person A is weighed 4 times and person B 16 times. Each set of readings has a mean of 70.0 kg. The figures are made up for this problem.

Someone reports 70.5 kg for each person instead of the mean. By how much does each person's sum of squared misses rise? Say what the difference between the two answers shows.

10. A paper's supplement gives, for one person's repeated body weights in kg, the sum of squared misses as a function of a candidate centre m. The formula is made up for this problem.

```
S(m) = 4m² - 560m + 19,601
```

How many readings were there? What was their mean? What is S at the mean, and what is the sample variance?

11. Here is a worked answer. Find the step that broke.

```
f(x) = -x² + 6x
f'(x) = -2x + 6, which is zero at x = 3
f(3) = -9 + 18 = 9
so the smallest value f ever takes is 9
```

12. Here is a worked answer for $f(x) = 2x^3 - 3x^2$ on the interval from -1 to 2. Find the step that broke.

```
f'(x) = 6x² - 6x = 6x(x - 1), zero at x = 0 and x = 1
f''(x) = 12x - 6
f''(0) = -6, so a local maximum, f(0) = 0
f''(1) = 6, so a local minimum, f(1) = -1
so over the interval, the largest value is 0 and the smallest is -1
```

13. Here is a worked derivation of the value of m that makes $\Sigma\,(x - m)^2$ smallest. Find the step that broke.

```
(x - m)² = x² - 2mx + m²
S(m) = Σ x² - 2m Σ x + m²
S′(m) = -2 Σ x + 2m
setting S′(m) = 0 gives m = Σ x
```

14. Here is a worked argument about a patient. The figures are made up. Find the step that broke.

```
her body weight rose for six weeks after she changed jobs, then fell
at week 6 her body weight was 82 kg, and its rate of change was zero
a rate of change of zero means an equilibrium
so 82 kg is her settling point
```

15. A press release says: “Weight loss rose with every dose we tested, so our top dose, 30 units, is the optimal dose.” The trial’s table, made up for this problem, gives the average weight loss at each dose.

dose (units)	weight loss (kg)
10	2.0
20	3.5
30	4.5

Decide what to compute, compute it, and say what your answer does not establish.

16. A colleague says: “The mean is the best single number to report for a patient’s repeated weighings. That is a mathematical fact.”

Here are five weighings of one person, made up for this problem. The last was taken in outdoor clothes: 70.2, 70.4, 70.3, 70.5 and 72.6 kg.

Decide what to compute, compute it, and say what your answer does not establish.

10 · Vectors as data rows, and the weighted sum

DEFINITION. A vector is an ordered list of numbers, called its entries. Its length is the number of entries it has. The order is part of the vector: (2, 5, 1) and (5, 2, 1) are different vectors.

One row of a data table is a vector, with one entry for each column, in the order of the columns.

Two vectors of the same length are added entry by entry. A vector is multiplied by a single number by multiplying every entry by that number. A single number used this way is called a scalar. Vectors of different lengths cannot be added.

A weighted sum of vectors v and w, with numbers c and d, is the vector cv + dw. The numbers c and d are its weights.

The dot product of two vectors of the same length, written a · b and said “a dot b”, is a single number. Multiply the entries in matching positions, then add the products. Two vectors of different lengths have no dot product. The dot product is a weighted sum of the entries of one vector, with the entries of the other as the weights. [79, 80, 81, 82, 83, 84, 85]

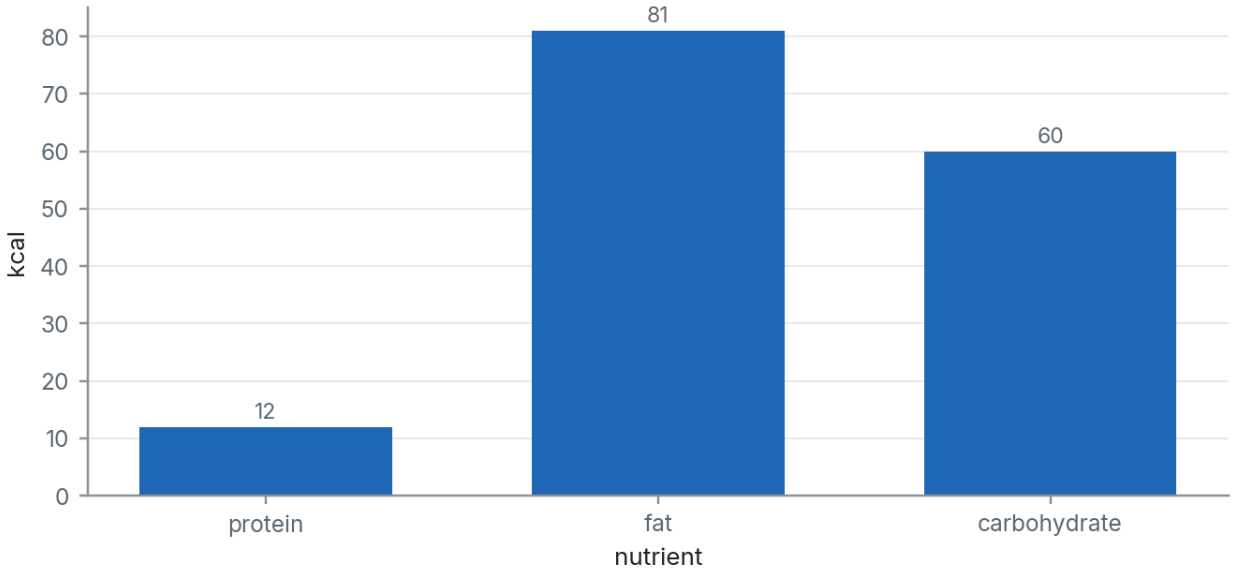
IN PLAIN TERMS. Write down a food’s grams of protein, fat and carbohydrate, always in that order. That list is a vector. The list (10, 30, 50) means 10 g of protein, 30 g of fat and 50 g of carbohydrate.

A gram of protein gives about 4 kcal. A gram of fat gives about 9, and a gram of carbohydrate about 4. So multiply each count of grams by its factor, and add.

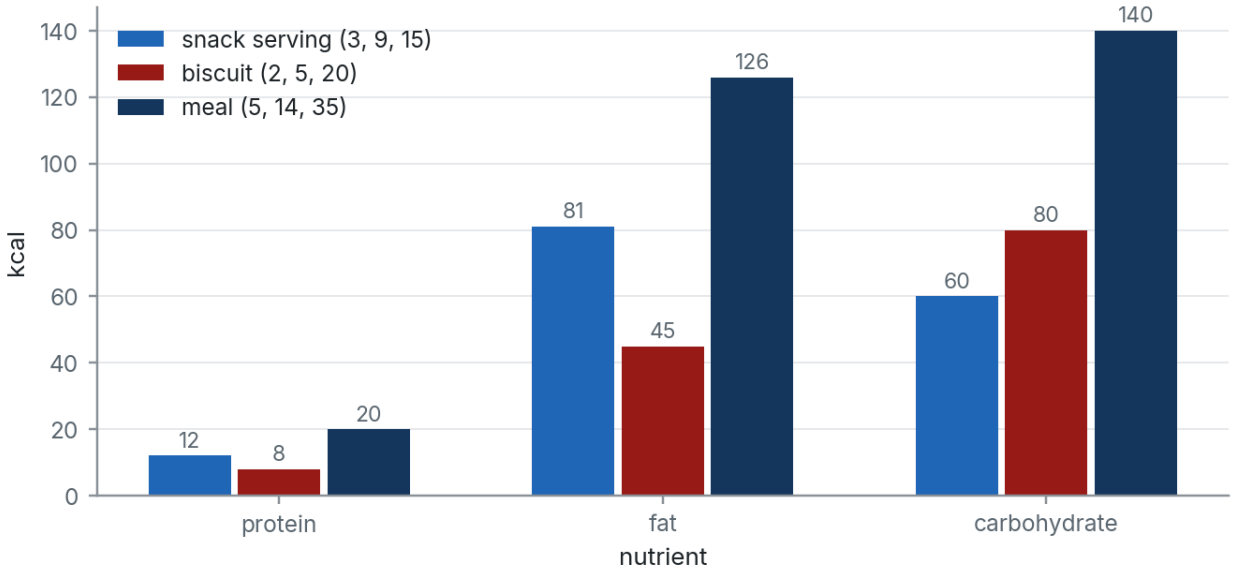
```
4 times 10 plus 9 times 30 plus 4 times 50 = 510
```

That multiply-and-add is called the dot product of the factor list (4, 9, 4) with the gram list (10, 30, 50). The answer is 510 kcal.

Each factor is a weight. “Weight” here means a multiplier, not body weight.



Where the 153 kcal in a 30 g serving of the made-up snack come from. Each bar is one term of the dot product (4, 9, 4) · (3, 9, 15): 12 kcal from protein, 81 from fat and 60 from carbohydrate.



The meal's energy, nutrient by nutrient. Each bar for the meal (5, 14, 35) is the snack serving's plus the biscuit's: 20 = 12 + 8, 126 = 81 + 45 and 140 = 60 + 80. So the meal's 286 kcal is 153 plus 133, whichever way it is worked out.

ILLUSTRATION. A clinic records four things about each patient, always in the same order. They are age in years, body weight in kg, height in cm and waist in cm. One patient's row, made up for this example, is a vector.

```
x = (40, 68, 162, 88)
```

Add a second patient's row to this one, and you get a list of four numbers that describes nobody. Multiply this row by 2 and you get a person of 80 years and 136 kg, which is not a doubled version of anyone.

One operation on these rows does mean something. Take the same patient a year later, with the new row y = (41, 65, 162, 84), also made up. Subtract entry by entry.

```
41 minus 40 = 1
65 minus 68 = -3
162 minus 162 = 0
84 minus 88 = -4
```

The change vector y − x is (1, −3, 0, −4).

A packaged snack's label gives its grams per 100 g. Take one made up for this example.

```
g = (10, 30, 50)
```

The entries are grams of protein, fat and carbohydrate in 100 g of the snack. A serving is 30 g, which is 0.3 of 100 g. Scale the vector by 0.3.

```
0.3 times 10 = 3
0.3 times 30 = 9
0.3 times 50 = 15
```

A serving holds (3, 9, 15): 3 g of protein, 9 g of fat and 15 g of carbohydrate.

Now the energy. The Food and Agriculture Organization (FAO) of the United Nations gives the Atwater general factors. They are 4.0 kcal/g for protein, 9.0 for fat and 4.0 for carbohydrate. Write them as a vector in the same order as the grams.

```
a = (4, 9, 4)
```

The energy in a serving is a · (3, 9, 15). Multiply matching entries and add.

```
4 times 3 = 12
9 times 9 = 81
4 times 15 = 60
12 plus 81 plus 60 = 153
```

A serving gives 153 kcal. The energy in the whole 100 g is a · g.

```
4 times 10 plus 9 times 30 plus 4 times 50 = 510
0.3 times 510 = 153
```

Scaling the grams by 0.3 scaled the energy by 0.3. Every product in the dot product was scaled by 0.3, so their sum was too.

Adding works the same way. A second item eaten with the snack, a biscuit, gives (2, 5, 20) per serving, also made up. The meal is the sum of the two vectors.

```
3 plus 2 = 5
9 plus 5 = 14
15 plus 20 = 35
```

The meal holds (5, 14, 35). Its energy can be found two ways, and they must agree.

```
4 times 5 plus 9 times 14 plus 4 times 35 = 286
4 times 2 plus 9 times 5 plus 4 times 20 = 133
153 plus 133 = 286
```

Dot the meal's vector once, or dot each item and add the energies. Both give 286 kcal.

Now read the weights. Add 1 g of fat to the serving and hold the other two entries fixed. Watch what the total does.

```
4 times 3 plus 9 times 10 plus 4 times 15 = 162
162 minus 153 = 9
```

You see the total rise by 9 kcal, the weight on fat. Each weight in a is the change in the total for one more unit of its own entry, with the others held fixed.

In India this is not only a textbook exercise. The Food Safety and Standards (Labelling and Display) Regulations, 2020, regulation 5(3)(e)(i), say a packaged food's label energy is calculated with fixed conversion factors. Among them are 4 kcal/g for carbohydrates and protein, and 9 kcal/g for fat. It lists others too, such as 2 kcal/g for dietary fibre. So the energy printed on a packet is a dot product of its declared grams with those factors.

Figures quoted in this illustration are from [86, 87].

WHERE THIS PICTURE BREAKS. The factors are general factors: one value for each nutrient, whatever the food. So a · g gives the energy a label would declare for a food whose only energy-giving parts are protein, fat and carbohydrate. It is not the energy a particular person absorbs from that food (Book 0, E2).

MUST KNOW POINTS FOR YOU.

- Before you take a dot product, line up the two vectors' orders.
- The misconception is that a weight tells you what happens when you change one thing in a real food. It tells you the change with every other entry held fixed. Swapping fat for sugar changes two entries at once. Write the whole change vector and dot that.
- Many different gram vectors give the same total energy. Never infer a food's make-up from its energy figure.
- Add or scale rows only when the sum, entry by entry, is itself an amount of something, such as the grams in a meal. Food grams add. Two patients' rows do not: their total age describes nobody. The difference between one patient's rows a year apart is a change, and that does mean something.
- Check a meal's energy in an app or a diet chart two ways. The two answers must agree. If they disagree by more than rounding, one item's grams or energy was entered wrong. Find it before you use the total.
- When you teach this, say "weights" for the multipliers and "body weight" for the person, every time.

EXERCISE 1 (interpretation). The FAO report gives the Atwater general factors in kilojoules as 17 kJ/g for protein, 37 kJ/g for fat and 17 kJ/g for carbohydrate. Write the energy E of a food, in kJ, as an equation in its grams of protein p, fat f and carbohydrate c. Say what each symbol stands for, with its unit. Then write the same equation as a dot product of two named vectors, and say which vector holds the weights.

PRACTICE. Fourteen problems on this technique, easiest first. Work them on paper before turning to the answers, which are in the appendix at the back under this section's name.

1. u = (2, 5, 1) and v = (3, −1, 4). Work out u + v and 3u.

2. With the same u = (2, 5, 1) and v = (3, −1, 4), work out the weighted sum 2u − 3v.

3. Work out u · v for u = (2, 5, 1) and v = (3, −1, 4). Then work out (4, 9, 4) · (0, 1, 0) and (4, 9, 4) · (1, 1, 1).

4. For each, say whether it can be worked out. Where it can, work it out.

1. (1, 2, 3) · (4, 5)
2. (1, 2) + (3, 4, 5)
3. 2.5 times (4, 0, −2)
4. (1, 2, 3) · (4, 5, 6)

5. A packaged food's label, made up for this problem, gives per 100 g: protein 7 g, fat 18 g, carbohydrate 66 g. Use the Atwater general factors of 4.0, 9.0 and 4.0 kcal/g (FAO 2003) to work out the energy in 100 g, in kcal.

6. Take the same food, (7, 18, 66) grams per 100 g. Work out its energy in kJ with the FAO factors of 17, 37 and 17 kJ/g. Then divide the kJ figure by the kcal figure from the last problem.

7. The food in the last two problems, (7, 18, 66) grams per 100 g, is eaten in a 25 g serving. Work out the grams in the serving, then the energy in the serving in kcal. Check the answer against the 454 kcal per 100 g.

8. FAO's Atwater system also gives alcohol a rounded factor of 7.0 kcal/g. A drink, made up for this problem, holds no protein, no fat, 10 g of carbohydrate and 14 g of alcohol. Write the grams and the factors as two vectors of length four, and work out the drink's energy in kcal.

9. A label, made up for this problem, gives 480 kcal per 100 g, 8 g of protein and 24 g of fat. The carbohydrate figure is smudged. Taking the energy as the dot product with (4, 9, 4), find the grams of carbohydrate.

10. A meal is one serving of each of three made-up foods. In grams of protein, fat and carbohydrate they are (4, 2, 30), (9, 6, 12) and (1, 8, 22). Work out the meal's energy in kcal in two ways, with the factors (4, 9, 4), and show that they agree.

11. Here is a worked answer. Find the step that broke.

```
a made-up label lists, per 100 g: fat 30 g, protein 10 g, carbohydrate 50 g
grams, as read off the label: (30, 10, 50)
factors for protein, fat, carbohydrate: (4, 9, 4)
energy = 4 times 30 plus 9 times 10 plus 4 times 50 = 410 kcal
```

12. Here is a worked answer for the energy in a 30 g serving of this section's made-up snack. Its grams per 100 g are (10, 30, 50). Find the step that broke.

```
the serving is 0.3 of 100 g
grams in the serving: 0.3 times (10, 30, 50) = (3, 9, 15)
factors for a serving: 0.3 times (4, 9, 4) = (1.2, 2.7, 1.2)
energy = 1.2 times 3 plus 2.7 times 9 plus 1.2 times 15 = 45.9 kcal
```

13. A food blog says: "Replace 10 g of fat in your recipe with 10 g of sugar. You cut 90 kcal, because fat carries 9 kcal a gram."

Decide what to compute, compute it, and say what your answer does not establish.

14. A colleague says: "These two snacks both give 510 kcal per 100 g, so nutritionally they are the same food."

Decide what to compute, compute it, and say what your answer does not establish.

11 · A matrix, and what multiplying by one does

DEFINITION. A matrix does for many rows at once what the dot product of the last section did for one. A matrix is a rectangular table of numbers, named by a capital letter. Its size is written rows by columns. A matrix with 3 rows and 2 columns is a 3 × 2 matrix, said "3 by 2". An entry is found by its row first and its column second.

A matrix A times a vector x, written Ax, is defined when A has as many columns as x has entries. The result is a vector with one entry for each row of A. Its entry for a given row is that row of A dotted with x. The same result is a weighted sum of the columns of A, with the entries of x as the weights.

A matrix A times a matrix B, written AB, is defined when A has as many columns as B has rows. If A is m × r and B is r × n, then AB is m × n. The entry of AB in row i and column j is row i of A dotted with column j of B.

A matrix times a scalar c, written cA, has every entry multiplied by c.

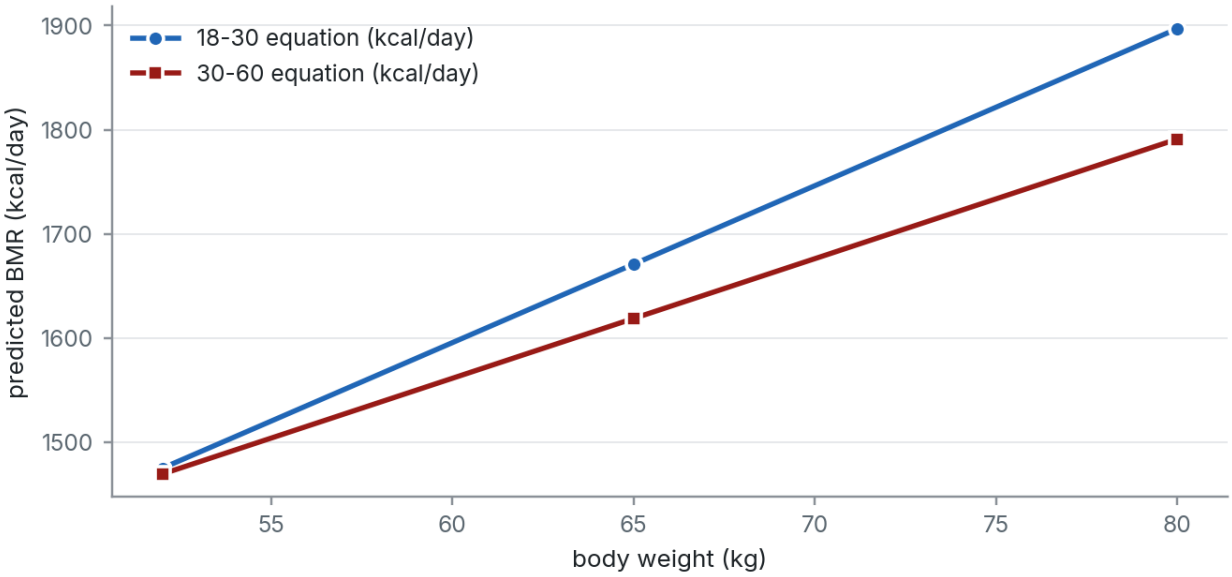
The fixed numbers that a formula multiplies by are its coefficients. A model written y = Xb says: each case's predicted value, an entry of y, is that case's row of the data matrix X dotted with one shared vector of coefficients b. [88, 89, 90, 91, 92, 93, 94, 95, 96, 97]

IN PLAIN TERMS. A matrix is a table of numbers with the labels taken off. In a data matrix, each row is one case and each column is one kind of measurement, always in the same place.

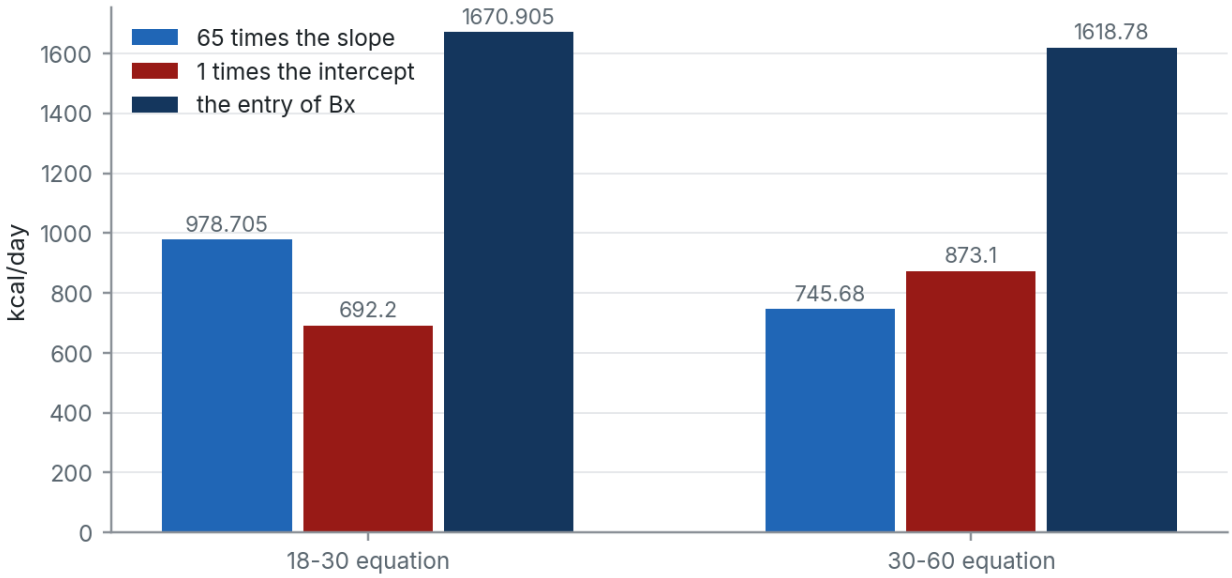
Multiplying a matrix by a vector does one dot product for each row.

Multiplying a matrix by a matrix does the same thing for several vectors at once.

The sizes have to fit. Each row of the first matrix is dotted with each column of the second. If they do not match, the product does not exist.



The two FAO/WHO/UNU equations for men, run on three men of 52, 65 and 80 kg (the product XC of the illustration below). The side axis does not start at zero, so the gap between the lines looks larger than it is.



Both equations run on a 65 kg man, read column by column (the product Bx of the illustration below): 65 lots of each equation's slope plus 1 lot of its intercept. The 18-30 equation gets more from body weight (978.705 against 745.68) and less from its intercept (692.2 against 873.1), and its total is higher, 1670.905 against 1618.78 kcal/day.

ILLUSTRATION. Book 1 met the basal metabolic rate (BMR): the energy spent lying at rest after a fast. Three United Nations bodies wrote a joint report on human energy requirements in 2004. The report predicts BMR from body weight with one straight line for each sex and age band. Here are the two rows for adult men, in kcal/day, as the report prints them.

men, age (years)	slope (kcal/day per kg)	intercept (kcal/day)
18-30	15.057	692.2
30-60	11.472	873.1

Each row says the same thing with its own numbers: BMR is the slope times body weight, plus the intercept. Take the labels off and keep the four numbers. That is a 2 × 2 matrix. Call it B.

Now you can run both equations on a man of 65 kg. That is the body weight the 2020 requirements of the Indian Council of Medical Research and National Institute of Nutrition (ICMR-NIN) use for an adult man. Write his vector as x = (65, 1). The 1 lines up with the intercept column, so the intercept is added once.

Row by row, each row of B dotted with x:

```
15.057 times 65 plus 692.2 times 1 = 1670.905
11.472 times 65 plus 873.1 times 1 = 1618.78
```

So Bx = (1670.905, 1618.78).

Column by column, 65 lots of the slope column plus 1 lot of the intercept column:

```
65 times 15.057 = 978.705
65 times 11.472 = 745.68
978.705 plus 692.2 = 1670.905
745.68 plus 873.1 = 1618.78
```

The same vector comes out.

Now three men, of 52, 65 and 80 kg; the 52 and 80 are made up. Put each man in a row, with his body weight and a 1: (52, 1), (65, 1) and (80, 1). That is a 3 × 2 data matrix. Call it X. Its rows are people.

To run both equations on all three men at once, put each equation in a column instead. This is the same four numbers as B, with its rows turned into columns. Call it C.

	18-30 equation	30-60 equation
multiplies body weight	15.057	11.472
multiplies the constant 1	692.2	873.1

X is 3 × 2 and C is 2 × 2. X has 2 columns and C has 2 rows, so the product XC is defined, and it is 3 × 2. Each entry is one man's row dotted with one equation's column.

Rounded to one decimal place, XC is this table.

body weight (kg)	18-30 equation (kcal/day)	30-60 equation (kcal/day)
52	1475.2	1469.6
65	1670.9	1618.8
80	1896.8	1790.9

Its rows are people and its columns are equations. Each column on its own is y = Xb: the data matrix X times one equation's coefficient vector b.

Now try the product the other way round, CX. C is 2 × 2 and X is 3 × 2. C has 2 columns but X has 3 rows, so CX does not exist.

Last, a scalar multiple. ICMR-NIN's 2020 brief note says the FAO/WHO/UNU equations run high for Indian adults. Its earlier committee had cut the BMR from those equations by 5%. The 2020 committee "further reduced the BMR by another 5 %". A cut of a fixed percentage multiplies every predicted BMR by one number. So it multiplies the whole matrix XC by one scalar.

The note does not say whether the second 5% comes off the original figure or off the figure already cut. The two readings give two scalars.

```
1 minus 0.05 minus 0.05 = 0.90
0.95 times 0.95 = 0.9025
```

Multiplying XC by 0.90 or by 0.9025 gives the two sets of reduced predictions.

Figures quoted in this illustration are from [16, 98].

WHERE THIS PICTURE BREAKS. The matrix product is exact arithmetic on the numbers it is given. It does not make the equations right for any one person. They were fitted to a large international set of measurements, and ICMR-NIN's note says that for Indians they can overestimate BMR by 10 to 12%.

For a real man, only the column for his own age band applies.

Every product comes out in kcal/day only because each slope meets a body weight in kg and each intercept meets the 1.

MUST KNOW POINTS FOR YOU.

- Check the sizes before you multiply. The first matrix needs as many columns as the second has rows, and the answer has the first one's rows and the second one's columns.
- Order matters. AB and BA are usually different, and one may not exist.
- The column of 1s is what carries the intercept. Leave it out of a data matrix, and the sizes no longer fit. Shorten the coefficients to make them fit, and every prediction silently loses the intercept.
- The note does not say whether the scalar is 0.90 or 0.9025. When you apply the cut, say which, and when you read a paper that applies it, look for which.
- Exact arithmetic is not an exact prediction. Quote a computed BMR as an estimate for a group, not as a patient's measured rate.
- Read y = Xb in a methods section as "each person's prediction is their row dotted with one set of coefficients". The multiplication tells you nothing about whether they were chosen well.

EXERCISE 1 (interpretation). A methods section says: "Predicted BMR was computed as y = Xb. Here X is an n × 2 matrix whose rows are (body weight in kg, 1), and b = (15.057, 692.2)."

Say what each symbol stands for, with its unit and size. Then say in one sentence what the equation claims.

PRACTICE. Fifteen problems on this technique, easiest first. Work them on paper before turning to the answers, which are in the appendix at the back under this section's name.

1. A is the matrix below.

	column 1	column 2	column 3
row 1	2	0	1
row 2	3	4	-1

What is the size of A? What is the entry in row 2, column 3? What is the entry in row 1, column 2?

2. With A as in the last problem, work out Ax for x = (1, 2, 3), row by row.

3. Work out the same Ax again, this time as a weighted sum of the columns of A.

4. Work out 3A, with A as in the first problem.

5. P has rows (1, 2) and (3, 4). Q has rows (0, 1) and (1, 0). Work out PQ and QP.

6. For each pair of sizes, say whether the product exists, and if it does, give its size.

1. a 3 × 2 matrix times a 2 × 4 matrix
2. a 2 × 4 matrix times a 3 × 2 matrix
3. a 2 × 2 matrix times a 2 × 1 matrix
4. a 1 × 3 matrix times a 3 × 1 matrix

7. FAO/WHO/UNU Table 5.2 gives these BMR equations for women, in kcal/day. For ages 18 to 30, the slope is 14.818 and the intercept 486.6. For ages 30 to 60, the slope is 8.126 and the intercept 845.6. ICMR-NIN's 2020 requirements use a body weight of 55.0 kg for an adult woman.

Write the equations as a 2 × 2 matrix, one equation in each row, and multiply it by the vector for a 55 kg woman. Say what each entry of the answer is.

8. Three women weigh 45, 55 and 70 kg; the 45 and 70 are made up. Write their data matrix X with rows (body weight, 1). Use the FAO/WHO/UNU equation for women aged 18 to 30, b = (14.818, 486.6) (Table 5.2), to work out y = Xb.

9. FAO/WHO/UNU say that multiplying a group's physical activity level (PAL) by its BMR gives its energy requirement. They treat a PAL of 1.75 or more as desirable. Take the vector of the two men's-equation BMRs for a 65 kg man from this section, (1670.905, 1618.78) kcal/day. Multiply it by the scalar 1.75.

10. Put the two FAO/WHO/UNU women's equations in the columns of a matrix D. The first column is (14.818, 486.6), for ages 18 to 30. The second is (8.126, 845.6), for ages 30 to 60. With X as in the problem before last, rows (45, 1), (55, 1) and (70, 1), work out XD. At which of the three body weights does the equation for ages 30 to 60 predict the higher BMR?

11. Here is a worked answer for a 65 kg man's BMR. It uses the FAO/WHO/UNU equation for men aged 18 to 30, with b = (15.057, 692.2). Find the step that broke.

```
his vector: (1, 65)
BMR = 15.057 times 1 plus 692.2 times 65
BMR = 45,008 kcal/day
```

12. Here is a worked answer for PQ, with P rows (1, 2) and (3, 4), and Q rows (5, 6) and (7, 8). Find the step that broke.

```
PQ is 2 × 2
multiply matching entries: 1 times 5, 2 times 6, 3 times 7, 4 times 8
PQ has rows (5, 12) and (21, 32)
```

13. Someone wants to apply a 10% cut to the FAO/WHO/UNU BMR equation for men aged 18 to 30. Here is their worked answer for a 65 kg man. Find the step that broke.

```
cut the equation by 10%: 0.9 times 15.057 = 13.5513
new equation: 13.5513 times body weight plus 692.2
BMR = 13.5513 times 65 plus 692.2 = 1573.0345 kcal/day
```

14. A colleague says: “ICMR-NIN says the FAO/WHO/UNU equations overestimate BMR in Indians by 10 to 12%. So multiply them by 0.88 to 0.90 to get the true figure.”

Decide what to compute, compute it, and say what your answer does not establish.

15. A clinic manager says: “Our spreadsheet computes every patient’s BMR with a single matrix multiplication, so the figures are exact.”

Decide what to compute, compute it, and say what your answer does not establish.

12 · A column that is a weighted sum of other columns

DEFINITION. Take a table whose rows are cases and whose columns are variables, as section 11 sets one up. Read each column as a vector, with one entry for each row.

A column y is a weighted sum of the columns a1, a2, ..., ak when there are fixed numbers w1, w2, ..., wk, called the weights, that make the following true on every row. The entry of y equals w1 times the entry of a1, plus w2 times the entry of a2, and so on up to wk times the entry of ak. The same weights must hold on every row.

A column of 1s counts as a column. So a column y that equals 2 times a, plus 5, on every row is a weighted sum of a and the column of 1s, with weights 2 and 5.

A column that is a weighted sum of others is redundant: once the other columns are known, it says nothing about any row that they did not already say.

The test for it treats the weights as unknowns. Each row gives one equation in them. Find the weights from as many rows as there are weights. With two weights, that is two equations in two unknowns, solved as in Book 0, C3.

With more, look for a row in which only one weight is still unknown. A row where every column but one is zero gives that weight at once. A row with the other weights already found gives the last one. This book solves more than two weights only this way.

Then put the weights into every other row. If every row holds, y is a weighted sum of those columns. If the equations have no solution, or a single leftover row fails, it is not. If the rows you picked repeat each other, so that their equations have infinitely many solutions, pick another row.

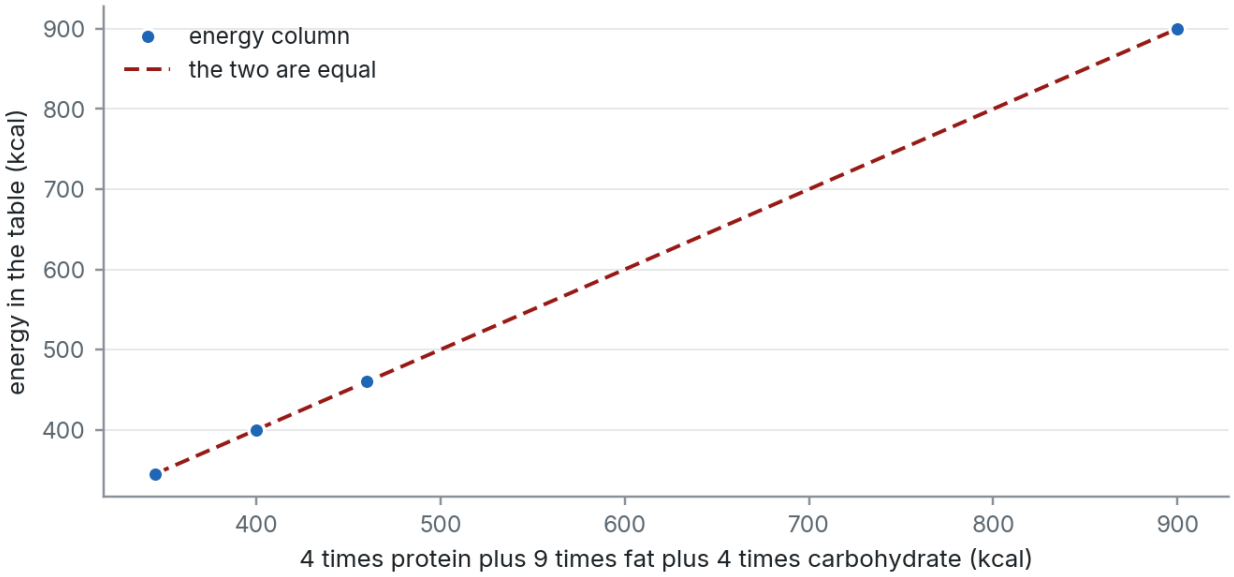
When a table holds a column beside the columns it is a weighted sum of, the table no longer fixes the weights on a weighted sum of all of them together. Weight can be moved from that column onto its parts, in proportion, in infinitely many ways, and every row's total stays the same.

A study that predicts an outcome from several columns (its predictors) chooses one weight for each from the rows. The weight on one column is an association, not a cause. It is the change in the prediction for one more unit of that column, with every other column held fixed, as in section 10.

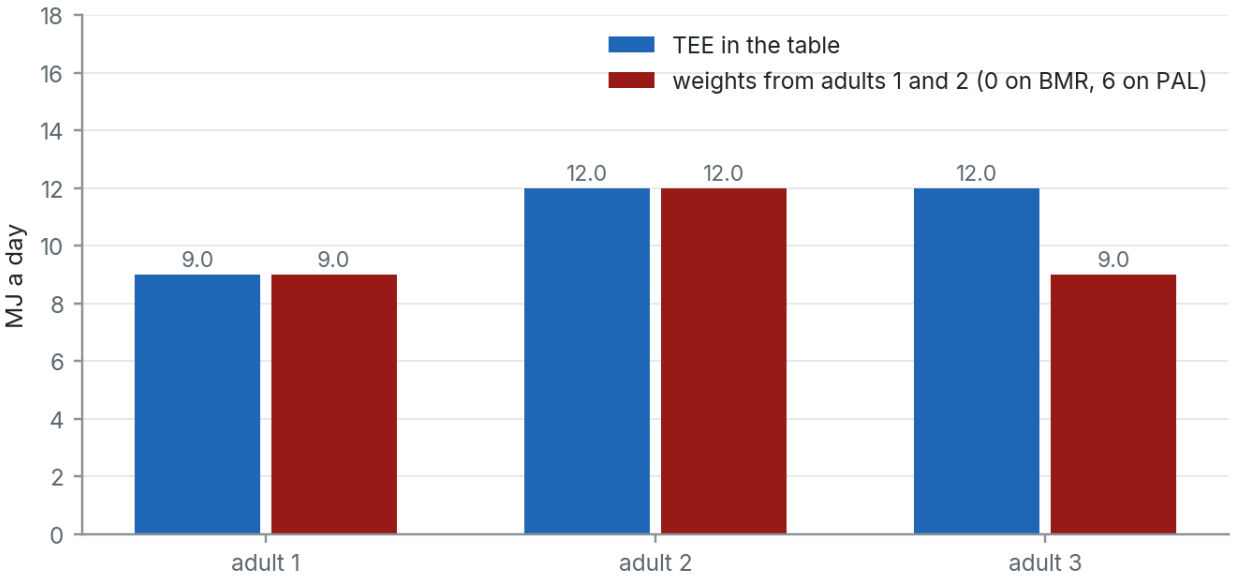
A column that is nearly, but not exactly, a weighted sum of others is a harder problem. The next rung takes it up. [99, 92, 100, 85, 101, 102, 37]

IN PLAIN TERMS. Picture an order sheet at a tiffin stall. Each row is one order. One column counts the idlis, one counts the dosas, and the last column is the amount paid.

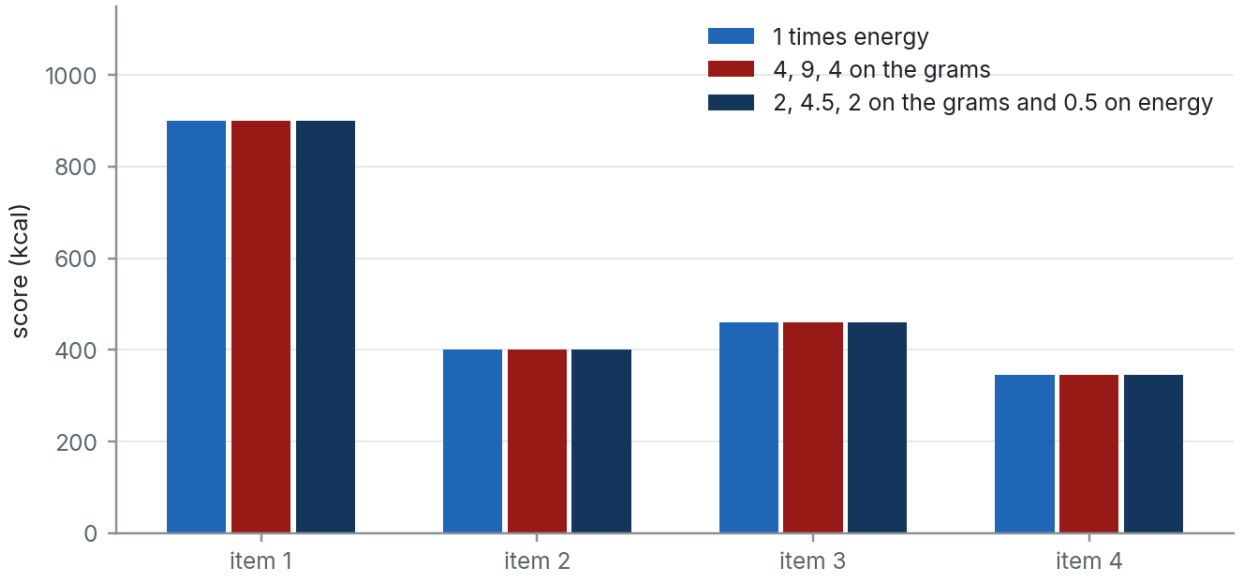
Suppose an idli costs 20 rupees and a dosa costs 50. Then on every row the amount paid is 20 times the idlis plus 50 times the dosas. The prices are the weights. The amount column adds nothing: cover it with your hand and you can still write it back in.



The four made-up items' energy against 4 times protein plus 9 times fat plus 4 times carbohydrate, both in kilocalories. Every point sits on the line where the two are equal, so the energy column says nothing the three gram columns do not.



The three made-up adults' TEE against the weights found from adults 1 and 2, 0 times BMR plus 6 times PAL, in megajoules a day. The weights fit the two adults they were found from and miss adult 3, whose TEE is 12.0, by 3.



Three made-up scores built from the four columns: 1 times energy; 4, 9 and 4 on the grams with none on energy; and 2, 4.5 and 2 on the grams with 0.5 on energy. They give the same total on every item, so the table cannot say which one a score comes from.

ILLUSTRATION. Here is a question that sounds reasonable and has no answer. “Among foods with the same protein, fat and carbohydrate, what goes with an extra 100 kilocalories?” Try to answer it from this table of four made-up items.

item	protein (g)	fat (g)	carbohydrate (g)	energy (kcal)
1	0	100	0	900
2	0	0	100	400
3	10	20	60	460
4	25	5	50	345

Look for two items that match in protein, fat and carbohydrate and differ in energy. There are none. The test shows why.

Write the energy column as unknown weights times the other three. Call the weight on protein p, on fat f and on carbohydrate c. On every row, energy must equal p times protein, plus f times fat, plus c times carbohydrate.

Row 1 has no protein and no carbohydrate, so it holds f alone: 100 times f is 900.

```
900 divided by 100 = 9
```

So f is 9. Row 2 holds c alone: 100 times c is 400.

```
400 divided by 100 = 4
```

So c is 4. Row 3 now has one unknown left, p. Put in the two weights you have.

```
20 times 9 plus 60 times 4 = 420
460 minus 420 = 40
40 divided by 10 = 4
```

So p is 4. Three rows went into finding three weights. Row 4 is the check.

```
4 times 25 plus 9 times 5 plus 4 times 50 = 345
```

The table says 345. The weights (4, 9, 4) hold on every row, so the energy column is a weighted sum of the other three.

Now go back to the question. Energy is 4 times protein plus 9 times fat plus 4 times carbohydrate. For it to rise while the grams stay fixed, one of those three parts would have to move. None can. The question asks about a row this table cannot contain.

Here is what that does to any sum built from all four columns. Take a made-up score for each item that equals its energy: 1 times energy, and no weight on the grams. Row 3 gives this.

```
1 times 460 = 460
```

Now put all the weight on the grams instead, and none on energy.

```
4 times 10 plus 9 times 20 plus 4 times 60 = 460
```

Now split it half and half: 2, 4.5 and 2 on the grams, and 0.5 on energy.

```
2 times 10 plus 4.5 times 20 plus 2 times 60 plus 0.5 times 460 = 460
```

All three give 460 on row 3, and they agree on every other row too. Handed only the table and the scores, nobody can say whether the score “comes from” energy or from the grams.

Now the other way round: a column that is fixed by two others and still fails the test.

Three United Nations bodies wrote the report *Human energy requirements* together in 2004. The report defines the physical activity level, PAL, as a person’s total energy expenditure for the day, TEE, divided by their basal metabolic rate, BMR. So, in the report’s words, “BMR times PAL is equal to TEE”.

Take three made-up adults.

adult	BMR (MJ/day)	PAL	TEE (MJ/day)
1	6.0	1.5	9.0
2	6.0	2.0	12.0
3	8.0	1.5	12.0

Test whether TEE is a weighted sum of BMR and PAL. Call the weights u and v. Rows 1 and 2 give two equations.

```
row 1: 6.0 times u plus 1.5 times v = 9.0
row 2: 6.0 times u plus 2.0 times v = 12.0
```

Subtract row 1 from row 2, the elimination move of Book 0, C3. The u terms cancel.

```
2.0 minus 1.5 = 0.5
12.0 minus 9.0 = 3.0
3.0 divided by 0.5 = 6
```

So 0.5 times v is 3.0, and v is 6. Put v back into row 1.

```
1.5 times 6 = 9
9.0 minus 9 = 0
```

So 6.0 times u is 0, and u is 0. Now check row 3, the row you did not use.

```
0 times 8.0 plus 6 times 1.5 = 9
```

The table says 12.0, not 9. Row 3 fails, so TEE is not a weighted sum of BMR and PAL.

Yet every TEE in the table is fixed by the other two columns.

```
6.0 times 1.5 = 9
6.0 times 2.0 = 12
8.0 times 1.5 = 12
```

The test finds weighted sums and only weighted sums. A column made by multiplying two others passes straight through it, and "not a weighted sum" must never be read as "carries something new".

Figures quoted in this illustration are from [86].

WHERE THIS PICTURE BREAKS. A real label also rounds its figures. The check then holds only to within the rounding. A row that misses by a fraction of a kilocalorie is not a sign of a separate column.

Add a column of 1s to the adults' test, and there are three weights for its three rows. The weights would then fit all three rows and leave nothing to check.

MUST KNOW POINTS FOR YOU.

- Find the weights from some rows, then check every other row. One row that fails settles it: the column is not a weighted sum.
- A table that carries energy beside protein, fat and carbohydrate, with energy worked out from the Atwater factors, carries one column too many. Refuse the question, or drop one column and say which one you dropped.
- The misconception is that a redundant column must look like one of the others. A column that looks like no other can still be a weighted sum of several. Comparing columns two at a time will not find it.

- The check needs rows to spare. With as many rows as weights, almost any column fits exactly.
- When a column is nearly but not exactly a weighted sum, this test cannot say whether that matters. Say so, and leave it to the next rung.
- When you hear that a model "found no effect of fat once calories were accounted for", ask how the calories were worked out. If they came from the grams, the model had no way to fix a weight for fat.

EXERCISE 1 (critique). A methods section says this.

> *Energy intake (kcal/day) was calculated from protein (P), fat (F) and carbohydrate (C) intakes as $E = 4P + 9F + 4C$. All four were entered together as predictors of weight change.*

Say what each symbol denotes, with its unit. Then say what the second sentence cannot deliver, and what the authors could have done instead.

PRACTICE. Fourteen problems on this technique, easiest first. Work them on paper before turning to the answers, which are in the appendix at the back under this section's name.

1. Take the columns a = (1, 2, 3) and b = (4, 0, 1).

Work out the column 3a + 2b, that is, 3 times a plus 2 times b, row by row.

2. Take a = (1, 2, 3), b = (1, 0, 2) and y = (5, 6, 13).

Find weights u and v with y equal to u times a plus v times b on every row, or show that there are none.

3. Take a = (2, 1, 4) and b = (1, 3, 0), and y = (7, 8, 8).

Is y equal to 2 times a plus 3 times b?

4. Take a = (2, 3, 1, 4), b = (1, 2, 5, 3) and y = (4, 7, 11, 9).

Test whether y is a weighted sum of a and b.

5. Take a = (1, 2, 4) and y = (8, 11, 17).

Test whether y is a weighted sum of a and the column of 1s.

6. A made-up packet label gives these figures for each 100 grams: protein 8 g, fat 12 g, carbohydrate 60 g, and energy 380 kcal.

Check whether the energy is the weighted sum of the grams with the Atwater general factors as weights.

7. A made-up table lists three items. None has any protein. For each 100 grams it gives the fat and the carbohydrate in grams. It gives the energy in kcal.

item	fat (g)	carbohydrate (g)	energy (kcal)
A	10	50	290

item	fat (g)	carbohydrate (g)	energy (kcal)
B	20	30	300
C	5	70	325

Find the weights that make energy a weighted sum of fat and carbohydrate. Check them. Give their unit. Then compare them with the Atwater factors.

8. Hall and colleagues (2012) write the energy balance equation as ES = EI − EO. Each term is an amount of energy per unit of time. ES is how fast the body's energy stores change. EI is the energy taken in. EO is the energy spent, which this book writes EE. Hall's ES is the rate that this book writes dES/dt.

Here are three made-up days, in kcal a day.

day	EI (kcal/day)	EE (kcal/day)	dES/dt (kcal/day)
1	2500	2400	100
2	2000	2200	-200
3	2300	2300	0

Without using the equation, find weights that make the dES/dt column a weighted sum of the EI and EE columns, and check them.

9. A table lists foods. It gives protein, fat and carbohydrate in grams, and energy in kilocalories. The energy was worked out with the Atwater general factors. An analyst finds that a score equals 0.01 times energy on every row.

Give two other sets of weights on the four columns (protein, fat, carbohydrate, energy) that give the same score on every row. Check both on the row with protein 10 g, fat 20 g, carbohydrate 60 g and energy 460 kcal.

10. Here is a worked answer. Find the step that broke.

```
a = (1, 2, 3, 5), b = (2, 1, 1, 1), y = (5, 4, 5, 9)
row 1: 1 times u plus 2 times v = 5
row 2: 2 times u plus 1 times v = 4
from row 2, v = 4 minus 2 times u; into row 1: u plus 8 minus 4 times u = 5, so u =
1
then v = 4 minus 2 = 2
the weights 1 and 2 fit, so y is a weighted sum of a and b, y = a + 2b
```

11. Here is a worked answer. Find the step that broke.

```
three made-up adults: (BMR 6.0, PAL 1.5, TEE 9.0), (6.0, 2.0, 12.0), (8.0, 1.5,
12.0), in MJ/day except PAL
test TEE as u times BMR plus v times PAL plus w times the column of 1s
solving the three rows gives u = 1.5, v = 6, w = -9
check: 1.5 times 6.0 plus 6 times 1.5 minus 9 = 9.0, and the other two rows also
hold
so TEE is a weighted sum of BMR, PAL and the column of 1s
```

12. Here is a worked answer. Find the step that broke.

```
a made-up label, per 100 g, printing energy as a whole number: protein 7.3 g, fat
11.6 g, carbohydrate 58.4 g, energy 367 kcal
Atwater weighted sum: 4 times 7.3 plus 9 times 11.6 plus 4 times 58.4 = 367.2
367.2 is not 367
so energy is not a weighted sum of the grams, and it carries information of its own
keep all four columns as separate predictors
```

13. A study reports this, and says its energy intake was worked out from the grams with the Atwater factors.

> *With energy intake, protein and carbohydrate in the model, fat intake had no effect on weight gain.*

The study's "effect" means the fitted weight on fat, not a cause.

Decide what to compute, compute it, and say what your answer does not establish.

14. A food composition table lists what is in each 100 grams of a food. Its notes say its carbohydrate is found "by difference". That is 100 minus water, protein, fat and ash (another column such tables report), all in grams. A colleague says:

> *Then the carbohydrate column is redundant. Drop it; we lose nothing.*

Decide what to compute, compute it, and say what your answer does not establish.

13 · A random variable and its distribution

DEFINITION. A random variable is a rule that gives a number to each outcome of a chance process. A random variable is written as a capital letter, such as X. P(X = x) is the probability that X takes the value x.

A random variable is discrete when the values it can take can be listed, one by one. Its probability distribution is the list of every value it can take, each paired with its probability. The values, or bands of values, listed must not overlap. Each probability is between 0 and 1, inclusive, and the probabilities add to 1.

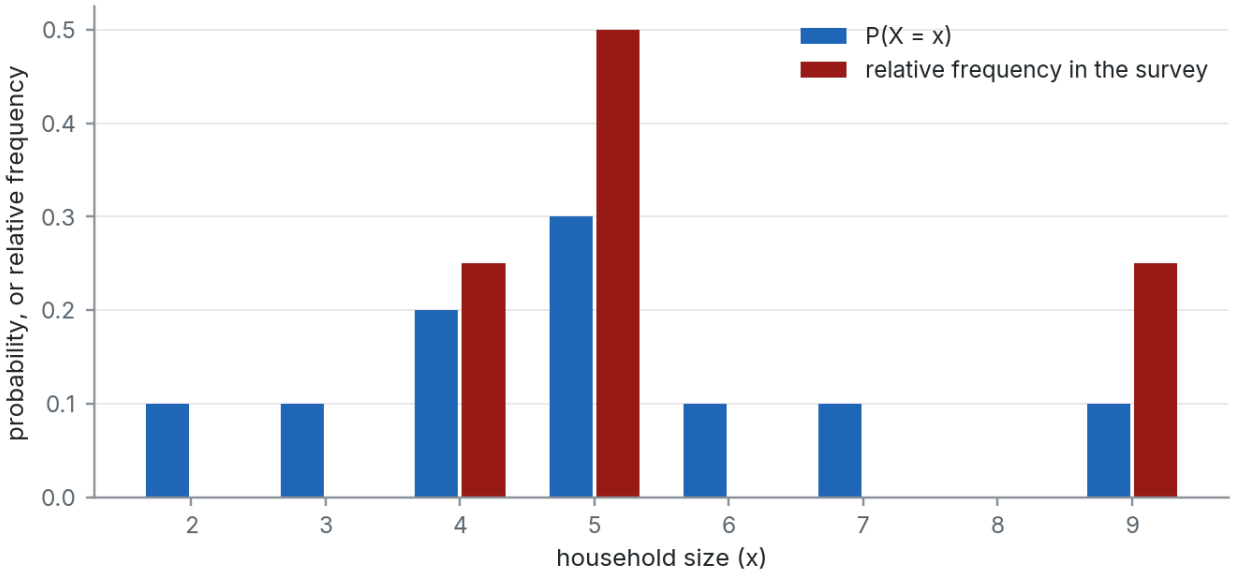
A random variable is continuous when it can take any value in an interval, like a weight measured with perfect precision, and its probabilities are areas under a curve. The probability of any one exact value is then 0. Its probability distribution is given by a probability density: a curve that never goes below zero and has a total area of 1 beneath it. The probability that X lands between a and b is the area under the probability density from a to b. The height of a probability density is a probability per unit of X, not a probability, and it can be larger than 1.

A probability distribution is a model of the chance process; a survey gives data. When the whole population is listed, as in the made-up village below, the model is known exactly. A frequency table of data, as in Book 0, D4, gives relative frequencies from the cases actually counted. Those estimate the probabilities, and differ from them from one sample to the next. [103, 104, 105, 106, 107, 108, 109, 110]

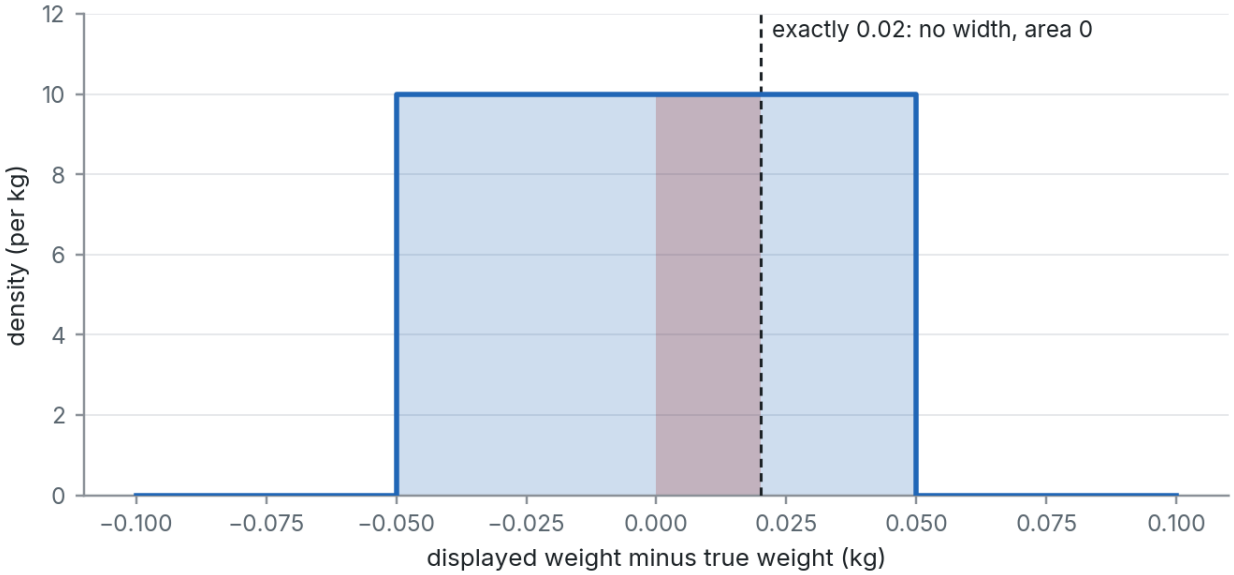
IN PLAIN TERMS. Toss two coins and count the heads. The count is 0, 1 or 2. That count is a random variable: a rule that turns each way the tosses can land into a number.

Its probability distribution lists each number beside its chance. Here that is 0.25 for no heads, 0.5 for one and 0.25 for two.

A list of chances is a model. A count of what happened in a survey is data.



The made-up village's household sizes. For each size, the first bar is the probability for one household picked at random and the second is the relative frequency in the survey of four. The survey puts 0.5 on 5 people where the probability is 0.3, and sees no household of 2, 3, 6 or 7 people, though each has probability 0.1.



The flat probability density for a scale's rounding error: 10 per kilogram from -0.05 to 0.05 kg, and zero outside. The whole rectangle has area 0.1 times 10, which is 1. The probability of an error between 0 and 0.02 kg is the strip over that range, 0.02 times 10, which is 0.2. The dashed line at exactly 0.02 has no width, so its probability is 0.

ILLUSTRATION. A made-up village has ten households. A survey visits four of them, picked at random, and finds households of 5, 9, 4 and 5 people. The note reads: "The probability that a household here has 5 people is 0.5." Two of the four had 5 people, and 2 divided by 4 is 0.5.

Here is the whole village. Its ten households hold 2, 3, 4, 4, 5, 5, 5, 6, 7 and 9 people.

The chance process is "pick one household at random", with each household equally likely, as in Book 0, D6. The random variable X is the rule "give the number of people in it".

Each household has a chance of 1 in 10.

```
1 divided by 10 = 0.1
```

Three households hold 5 people, so P(X = 5) is 3 times 0.1.

```
3 times 0.1 = 0.3
```

Here is the probability distribution of X, and beside it the relative frequencies from the survey's four households.

```
household size (x) households P(X = x) relative frequency in the survey
2    1    0.1    0
3    1    0.1    0
4    2    0.2    0.25
5    3    0.3    0.5
6    1    0.1    0
7    1    0.1    0
9    1    0.1    0.25
```

Check the two rules. Every probability is between 0 and 1. Then add them.

```
0.1 plus 0.1 plus 0.2 plus 0.3 plus 0.1 plus 0.1 plus 0.1 = 1.0
```

They add to 1.

The probability of a household of 6 or more is the sum over 6, 7 and 9.

```
0.1 plus 0.1 plus 0.1 = 0.3
```

Before you write the survey's 0.5 into a report, set it beside P(X = 5), which is 0.3. The survey's 0.5 is a relative frequency from four households. It estimates 0.3, and misses it.

A clinic scale shows weight to 0.1 kg. A person whose true weight is anywhere from 69.95 kg to 70.05 kg sees 70.0 on it. Let X be the displayed weight minus the true weight, in kilograms. That is the error the rounding makes. X can be any number from −0.05 to 0.05, so it is continuous.

Say nothing makes one part of that range more likely than another. Then the probability density is flat across a width of 0.1 kg, and zero outside it. Its total area must be 1, so its height is 1 divided by the width.

```
1 divided by 0.1 = 10
```

The height is 10 per kilogram.

The probability that the error lies between 0 and 0.02 kg is the area of a rectangle, 0.02 kg wide and 10 per kilogram high.

```
0.02 times 10 = 0.2
```

Now ask for the probability that the error is exactly 0.02. That rectangle has no width, so its area is 0.

WHERE THIS PICTURE BREAKS. The village is made up, and the distribution is exact only because the whole village was listed.

The flat density is a model chosen for want of anything better.

MUST KNOW POINTS FOR YOU.

- A relative frequency from data is not the probability. It is an estimate of it.
- Before you use a table that a paper calls a distribution, check it. Every probability is between 0 and 1, and they add to 1. No two rows overlap: bands of "under 60 kg" and "50 to 70 kg" count the same people twice.
- For a continuous measurement, the chance of any single exact value is zero. Ask for the chance of a range, such as 69.95 to 70.05 kg, never of "exactly 70 kg".
- Every probability you work out from a distribution is only as good as the distribution. Say so in the same sentence, or get data that shows the shape.
- A zero in a small sample does not make a value impossible. Say "not seen in this sample", never "probability zero", unless you hold the whole list.
- When you give a journalist a chance, say whether it is a probability from a model or a relative frequency from a sample, and how big the sample was.

EXERCISE 1 (teaching). A journalist has three minutes. They have read "the probability that a household has five members is 0.5" in a survey report of four households, and want to print it.

Explain the difference between a probability and a relative frequency from a survey. Give them one sentence they can quote.

PRACTICE. Fifteen problems on this technique, easiest first. Work them on paper before turning to the answers, which are in the appendix at the back under this section's name.

1. X takes the values 0, 1, 2 and 3, with probabilities 0.1, 0.3, 0.4 and 0.2.

Is this a probability distribution?

2. X takes the values 1, 2, 3 and 4. $P(X = 1)$ is 0.25, $P(X = 2)$ is 0.35 and $P(X = 4)$ is 0.15.

Work out $P(X = 3)$.

3. Toss two fair coins. Let X be the number of heads.

List the outcomes, and write out the probability distribution of X.

4. X takes the values 0, 1, 2, 3 and 4, with probabilities 0.1, 0.2, 0.4, 0.2 and 0.1.

Work out $P(X \geq 3)$, the probability that X is 3 or more. Then work out $P(1 \leq X \leq 3)$, the probability that X is from 1 to 3. Then work out $P(X = 0)$ by the complement.

5. Three tables are offered as probability distributions for a variable taking the values 1, 2 and 3.

```
table P(X = 1) P(X = 2) P(X = 3)
a    0.3    0.3    0.3
b    0.5    -0.1    0.6
c    0.2    0.5    0.3
```

Which are probability distributions? For each that is not, name the rule it breaks.

6. A continuous random variable X has a flat probability density from 0 to 4, and zero density outside that range.

Work out the height of the density. Then work out $P(1 \leq X \leq 2.5)$ and $P(X = 3)$.

7. A continuous random variable X has a probability density that is zero outside the range from 0 to 2. Inside it, the density rises in a straight line, from 0 at $x = 0$ to 1 at $x = 2$.

Check that the total area is 1. Then work out $P(X \leq 1)$.

8. A made-up ward has 20 patients. Eight stayed 2 days, six stayed 3 days, four stayed 4 days and two stayed 7 days. Pick one patient at random. Let X be their length of stay, in days.

Write out the probability distribution of X. Then work out the probability of a stay of 4 days or more.

9. A scale shows weight to the nearest 0.2 kg. Let X be the displayed weight minus the true weight, in kilograms. Model X as spread evenly, with a flat probability density, across its whole interval.

Work out the interval of values X can take and the height of the density, with its unit. Then work out the probability that the error is more than 0.05 kg away from zero, in either direction.

10. Go back to the made-up ward of 20 patients: eight stayed 2 days, six 3 days, four 4 days and two 7 days. A student picks five patients at random and finds stays of 2, 2, 7, 3 and 2 days.

Work out the relative frequency of a 2-day stay in the student's five. Set it beside $P(X = 2)$ for the ward, and say which is the model and which is the estimate.

11. Here is a worked answer. Find the step that broke.

```
a scale rounds to the nearest 0.1 kg; the error X is modelled as flat from -0.05 to
0.05 kg
height of the density = 1 divided by 0.1 = 10
so P(X = 0.02) = 10
that is, an error of exactly 0.02 kg is very likely
```

12. Here is a worked answer. Find the step that broke.

```
the made-up ward: 8 patients stayed 2 days, 6 stayed 3 days, 4 stayed 4 days, 2
stayed 7 days
a student samples five patients at random: stays 2, 2, 7, 3, 2
no patient in the sample stayed 4 days
so P(X = 4) = 0 for this ward: a 4-day stay does not happen here
```

13. Here is a worked answer. Find the step that broke.

```
a report gives weights in three bands: under 60 kg, 0.3; 50 to 70 kg, 0.5; over 70
kg, 0.3
P(under 70 kg) = P(under 60 kg) plus P(50 to 70 kg)
0.3 plus 0.5 = 0.8
so the probability of weighing under 70 kg is 0.8
```

14. A clinic note says this. The counts are made up.

> *In our survey, 12 of 40 patients weighed between 80 and 90 kg. So the probability that the next patient weighs between 80 and 90 kg is 0.3.*

Decide what to compute, compute it, and say what your answer does not establish.

15. PAL is the physical activity level: total energy expenditure divided by basal metabolic rate. The FAO/WHO/UNU report on energy needs says this about it.

> *The PAL values that can be sustained for a long period of time by free-living adult populations range from about 1.40 to 2.40.*

A slide then says:

> *So a randomly chosen adult has a 50 per cent chance of a PAL above 1.90.*

Decide what to compute, compute it, and say what your answer does not establish.

14 · Expectation

DEFINITION. Let X be a discrete random variable, as in section 13, taking the values x1, x2, ..., xk. Its expected value, or expectation, is written E[X]. It is each value times its probability, all added up: x1 times P(X = x1), plus x2 times P(X = x2), and so on up to xk times P(X = xk). In the sigma notation of section 1, E[X] = Σ x P(X = x), the sum running over every value x that X can take. It carries the unit of X.

E[X] is a probability-weighted average. When every listed case is equally likely, as with the households of the village below, E[X] is the ordinary mean of Book 0, D5. In general it is not. It need not be a value X can take.

It is the long-run average: repeat the chance process many times, and the average of the values X takes settles towards E[X]. That is the law of large numbers. E[X] is a property of the probability distribution, a parameter in the sense of Book 0, D6. The mean of a sample of observed values is a statistic. It estimates E[X] and differs from it from one sample to the next.

Two rules follow from the definition. The first: for fixed numbers a and b, E[aX + b] = aE[X] + b. Each value x becomes ax + b, with the same probability. Multiply out, as in Book 0, C1.

```
Σ (ax + b) P(X = x) = a Σ x P(X = x) + b Σ P(X = x) = aE[X] + b
```

The last step uses the fact that the probabilities add to 1.

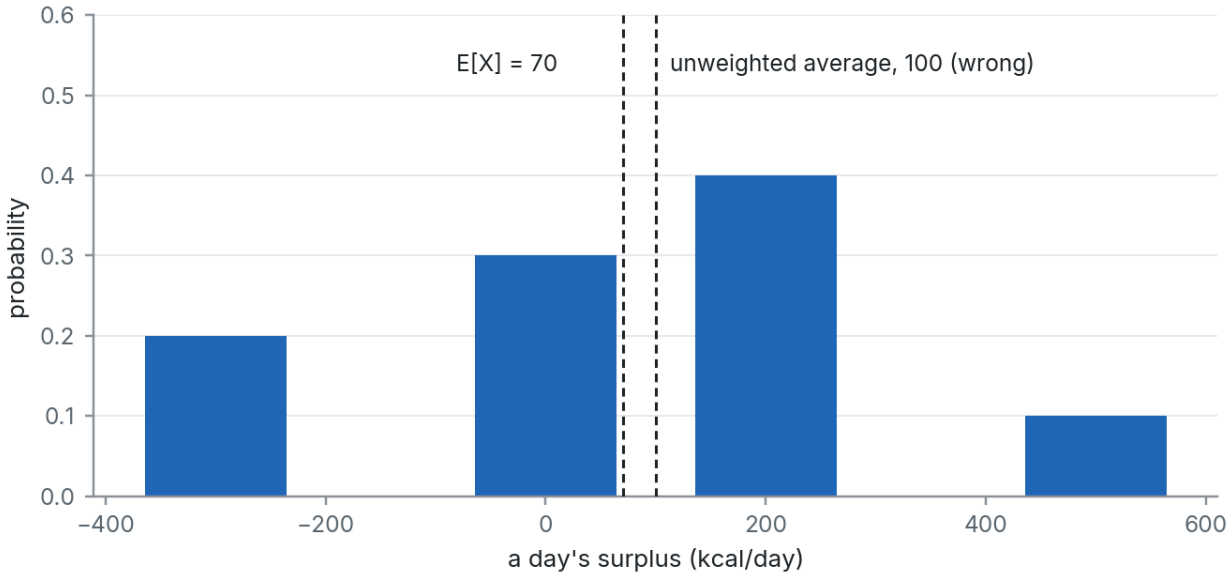
The second: for any two random variables X and Y, E[X + Y] = E[X] + E[Y]. A random variable gives a number to each outcome, so E[X] is also the sum, over outcomes, of X's number times the outcome's probability. Outcomes that give the same number group back into Σ x P(X = x). X + Y gives each outcome X's number plus Y's. So its sum splits into the sum for X plus the sum for Y. No step asked whether X and Y are independent. Independent means that knowing one tells you nothing about the other, the idea of Book 0, D2 carried to random variables. Together the rules give E[aX + bY] = aE[X] + bE[Y].

Neither rule reaches products, ratios or squares. E[XY] is not in general E[X] times E[Y], and E[X/Y] is not in general E[X] divided by E[Y]. For those, go back to the distribution.

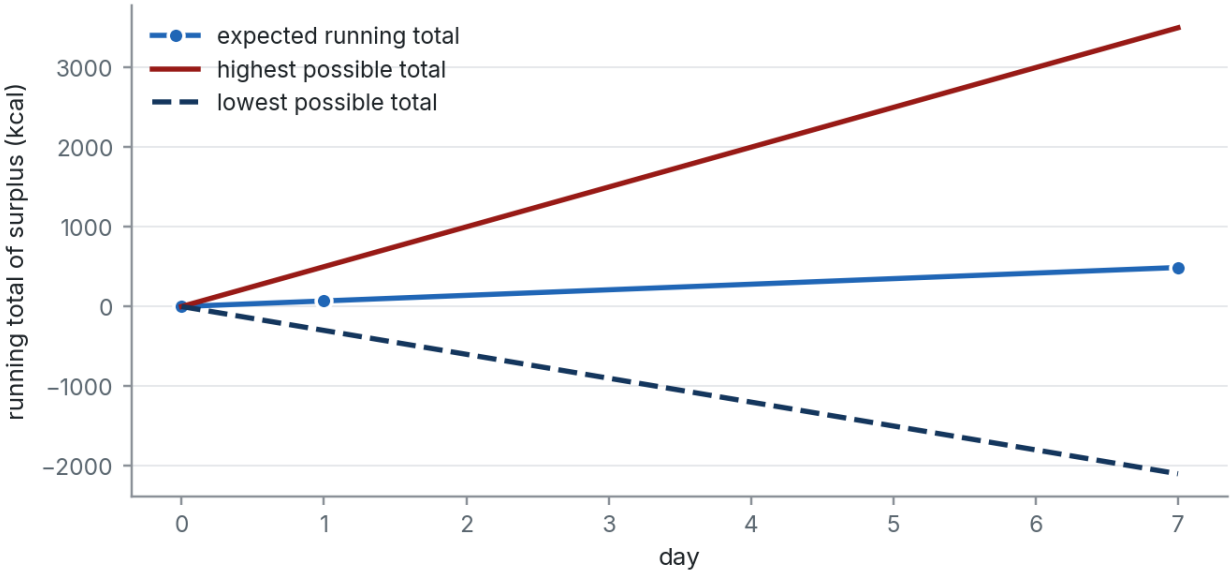
In papers on energy balance, E with a label, as in EI, EE or ES, is an energy quantity: a store in kcal, or a rate in kcal a day. E followed by square brackets around a random variable, E[X], is expectation. Some books write E(X) with round brackets. [111, 112, 113, 114, 115, 116, 117, 118, 37]

IN PLAIN TERMS. Take a chance process that gives a number, and imagine running it a very large number of times. The average of all those numbers is the expected value.

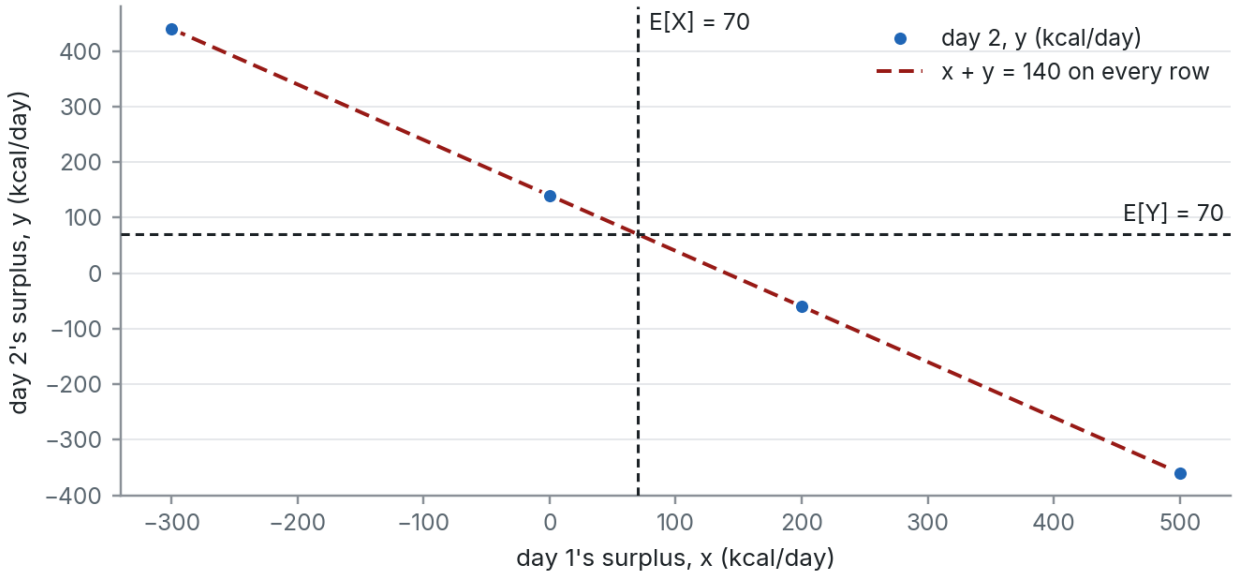
Multiply each value by its chance and add.



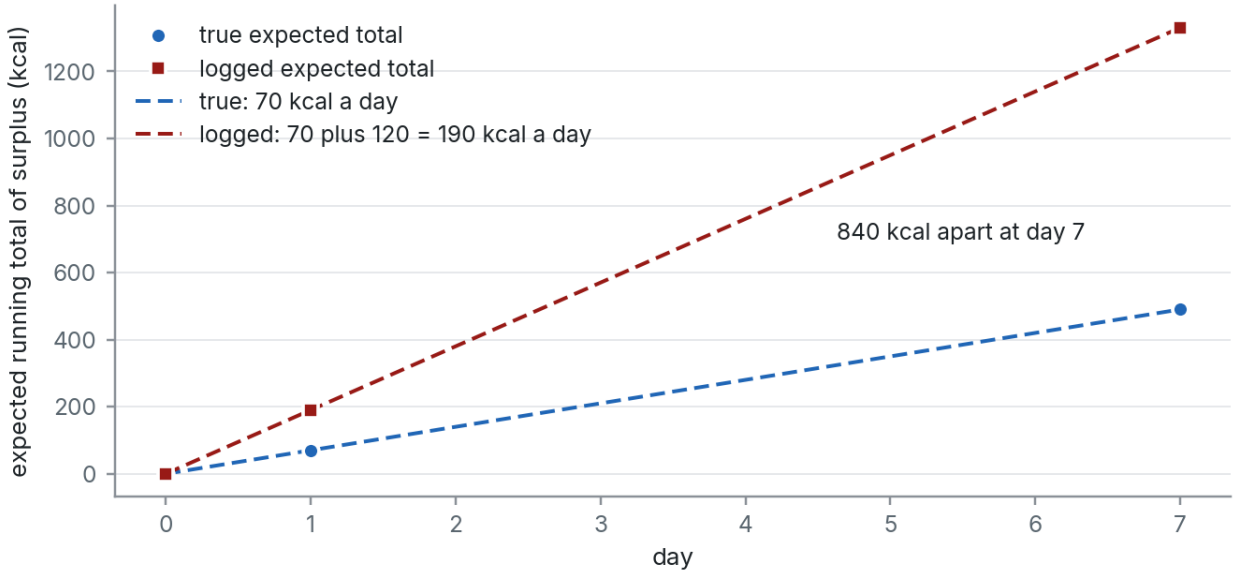
The made-up distribution of a day's energy surplus, in kilocalories a day. The expected value, 70, is where the four bars would balance, each weighted by its probability. No bar sits at 70. Averaging the four values as if each were equally likely gives 100, pulled right by the 500 that has probability only 0.1.



The expected running total of surplus over a week grows by 70 kilocalories a day, to 490 by day 7. The possible totals fan out much wider, from seven days at minus 300 (minus 2,100) to seven days at 500 (3,500), so 490 is not what any one week will show.



Day 2's surplus is always 140 minus day 1's, so the four possible pairs all lie on the line x + y = 140, as dependent as two days can be. The expected values, 70 for each day, meet on the same line: E[X + Y] is 70 plus 70, which is 140, and the sum rule needs no independence.



A fixed 120 kcal added to every logged day moves the expected surplus from 70 to 190 kcal a day. Over the week the expected log reaches 1,330 kcal against a true 490: 840 apart, which is 7 times 120. More days of logging widen the gap; they never close it.

ILLUSTRATION. Suppose a person's energy surplus on a day, intake minus expenditure as in Book 1, section 5, is a random variable X, in kilocalories a day. The distribution below is made up.

surplus x (kcal/day)	P(X = x)
-300	0.2
0	0.3

surplus x (kcal/day)	P(X = x)
200	0.4
500	0.1

The move you will reach for first is to average the four values, as if each were equally likely.

```
-300 plus 0 plus 200 plus 500 = 400
400 divided by 4 = 100
```

That gives 100, and it is wrong. Weight each value by its probability instead.

```
-300 times 0.2 = -60
0 times 0.3 = 0
200 times 0.4 = 80
500 times 0.1 = 50
-60 plus 0 plus 80 plus 50 = 70
```

E[X] is 70 kcal a day. It is where the four bars would balance, each weighted by its probability. You will never see a day with a surplus of 70 in this model.

Now a week. Let W be the surplus over seven days: X1 for day 1, plus X2 for day 2, and so on to X7. If each day has this same distribution, each has expected value 70. The sum rule gives the week.

```
7 times 70 = 490
```

E[W] is 490 kcal.

So 490 is the long-run average of weeks. It is not what any one week will show.

Does the sum rule need the days to be independent? Try a pair that is as tied together as possible. Say that on day 2 this person's surplus is always 140 minus day 1's. A big day is followed by a small one. Call day 2's surplus Y. Each value of X fixes Y.

day 1, x (kcal/day)	day 2, y (kcal/day)	probability
-300	440	0.2
0	140	0.3
200	-60	0.4
500	-360	0.1

Work out E[Y] from its own values and probabilities.

```
440 times 0.2 plus 140 times 0.3 plus -60 times 0.4 plus -360 times 0.1 = 70
```

E[Y] is 70 too. On every row, X plus Y is 140, so E[X + Y] is 140. The sum rule gives the same.

```
70 plus 70 = 140
```

The two days could hardly be more dependent, and the rule still holds.

Now a fixed shift. Say an app adds a fixed 120 kcal to every day's logged intake, a systematic error as in Book 0, D7. The logged surplus is X + 120. By the first rule its expected value is E[X] + 120.

```
70 plus 120 = 190
7 times 190 = 1330
```

Over a week the log expects 1,330 kcal against a true 490. Logging more weeks will not close that gap.

Go back to the made-up village of section 13: ten households of 2, 3, 4, 4, 5, 5, 5, 6, 7 and 9 people. Pick one at random, and let X be its size. Each household has probability 0.1, so E[X] is 0.1 times the sum of the sizes.

```
2 plus 3 plus 4 plus 4 plus 5 plus 5 plus 5 plus 6 plus 7 plus 9 = 50
0.1 times 50 = 5.0
```

The survey's four households held 5, 9, 4 and 5 people.

```
5 plus 9 plus 4 plus 5 = 23
23 divided by 4 = 5.75
```

The sample mean is 5.75. It is a statistic that estimates E[X], and it misses by 0.75. A bigger random sample would tend to land closer.

WHERE THIS PICTURE BREAKS. The surplus distribution is made up, and it is the same on every day only because this model says so. A real person's daily surplus has no known distribution. It also drifts as their weight and habits change, and then "7 times 70" no longer describes the week. The sum rule still holds with seven different expectations.

The village is made up.

MUST KNOW POINTS FOR YOU.

- The expected value is not what any one person or day will show. Never tell a patient "you can expect to lose 4 kg" as if it were their outcome. Say what happened on average, and how widely people differed.
- Averaging the possible values without their probabilities can be trusted only when every value is equally likely. Weight each value by its probability, every time.
- The expected value of a sum is the sum of the expected values, with no condition of independence. Do not refuse the sum because the days are dependent.
- The rules stop at sums and fixed multiples. The expected BMI is not the expected weight divided by the square of the expected height. For a product, a ratio or a square, go back to the distribution and work value by value.
- A fixed error of b in every reading moves the expected value by exactly b. Find the size of the shift against a known reference, as in Book 0, D7, and subtract it.

- E[X] belongs to the model. Before you quote a paper's "expected value", ask whether it was computed from a stated distribution or averaged from data.
- In an energy-balance paper, check which E you are reading.

EXERCISE 1 (interpretation). A methods section contains these two lines.

> *E(X) = Σ x_i P(X = x_i), where X is the daily energy surplus in kcal/day.*
>
> *Mean EI was 2,400 kcal/day.*

Read the first line aloud in words, symbol by symbol, with units. Then say which E in the two lines is energy and which is expectation, and how you can tell.

PRACTICE. Fourteen problems on this technique, easiest first. Work them on paper before turning to the answers, which are in the appendix at the back under this section's name.

1. X takes the values 1, 2 and 3, with probabilities 0.2, 0.5 and 0.3.

Work out E[X].

2. X takes the values −2, 1 and 4, with probabilities 0.5, 0.25 and 0.25.

Work out E[X]. Is it a value X can take?

3. E[X] is 3.

Work out E[2X + 5], E[−X] and E[X − 3].

4. E[X] is 4 and E[Y] is −1. Nothing is known about whether X and Y are independent.

Work out E[X + Y], E[X − Y] and E[3X − 2Y].

5. X takes only the values 0 and 10. E[X] is 2.5.

Work out P(X = 10).

6. The FAO/WHO/UNU report on human energy requirements gives this example. A man has a PAL of 1.75 and a mean BMR of 7.10 MJ/day. The report multiplies them to get his mean energy requirement.

Treat BMR as a random variable with expected value 7.10 MJ/day, and PAL as fixed at 1.75. TEE is PAL times BMR. Work out E[TEE], and compare it with the report's figure of 12.42 MJ/day.

7. In a made-up model, a person's daily energy surplus has expected value −150 kcal/day on every day. Their days are not independent: a low day tends to be followed by a high one.

Work out the expected surplus over 7 days and over 28 days.

8. In a made-up model, a person's true daily surplus has expected value 30 kcal/day. Their food app adds a fixed 120 kcal to every day's logged intake.

Work out the expected logged surplus for a day and for a week. Then work out how far the week's expected logged surplus is from the true one.

9. Go back to the made-up village of ten households with 2, 3, 4, 4, 5, 5, 5, 6, 7 and 9 people. Its E[X] for one household picked at random is 5.0 people. A second survey picks five households at random and finds 4, 7, 5, 5 and 9 people.

Work out the second survey's mean. Say which number is the parameter and which the statistic, and what would tend to happen to the statistic with a much bigger sample.

10. Here is a worked answer. Find the step that broke.

```
a day's surplus X is -300, 0, 200 or 500 kcal, with probabilities 0.2, 0.3, 0.4 and
0.1
E[X] = (-300 plus 0 plus 200 plus 500) divided by 4
= 400 divided by 4 = 100
so the expected surplus is 100 kcal a day
```

11. Here is a worked answer. Find the step that broke.

```
each day's surplus has expected value 70 kcal a day
the week's expected surplus is the average of the seven days' expected values
(70 plus 70 plus 70 plus 70 plus 70 plus 70 plus 70) divided by 7 = 70
so the expected surplus for the week is 70 kcal
```

12. Here is a worked answer. Find the step that broke. BMI, from Book 0, B6, is weight in kilograms divided by the square of height in metres.

```
two made-up adults, each picked with probability 0.5: 50 kg and 1.5 m, or 90 kg and
1.8 m
E[weight] = 0.5 times 50 plus 0.5 times 90 = 70
E[height] = 0.5 times 1.5 plus 0.5 times 1.8 = 1.65
E[BMI] = E[weight] divided by E[height] squared = 70 divided by 2.7225 = 25.7
```

13. A programme leaflet says this.

> *The expected weight loss on our programme is 4 kg. So you will lose 4 kg.*

Here is the leaflet's model, which is made up. The loss is 0 kg with probability 0.3, 4 kg with probability 0.4, and 8 kg with probability 0.3. Decide what to compute, compute it, and say what your answer does not establish.

14. A health talk says this.

> *Eating just 100 kcal a day more adds up to 36,500 kcal in a year, and all of it is stored.*

Decide what to compute, compute it, and say what your answer does not establish.

15 · Variance, and why independent errors add as squares

DEFINITION. In section 14 you saw where a random variable sits on average. The variance says how far it strays.

The variance of X, written Var(X) and said "the variance of X", is the expected value of the squared deviation of X from E[X]. In symbols:

```
Var(X) = E[(X - E[X])²]
```

For a random variable with a list of values, that is the sum, over every value x, of $(x - E[X])^2$ times the probability of x. The standard deviation of X, written with the Greek letter sigma, σ, is the square root of Var(X). It carries the unit of X. The variance carries that unit squared. Book 0, D5 found a sample's standard deviation by dividing by n − 1. That is a statistic. It estimates σ, as a sample mean estimates E[X].

For fixed numbers a and b, the variance of aX + b is a^2 times the variance of X. The standard deviation of aX + b is σ times the size of a, whatever the sign of a.

```
Var(aX + b) = a² Var(X)
```

Two random variables are independent when every event about one is independent, in the sense of Book 0, D2, of every event about the other. For independent X and Y, the variance of their sum is the sum of their variances, and so is the variance of their difference.

```
Var(X + Y) = Var(X) + Var(Y)
Var(X - Y) = Var(X) + Var(Y)
Var(aX + bY) = a² Var(X) + b² Var(Y)
```

This book checks the rule on the dal example below and does not prove it. The proof needs one more fact, that E[XY] = E[X] times E[Y] when X and Y are independent.

Standard deviations do not add.

For n independent random variables, each with variance σ^2, the sum has variance $n\sigma^2$ and standard deviation σ times the square root of n. Their mean has standard deviation σ divided by the square root of n, which is the square-root law of Book 0, D6.

Without independence the adding rule need not hold. A correction term, the covariance, enters, and a later book, on probability and statistical inference, teaches it. [119, 120, 121, 122, 38, 123, 124, 125]

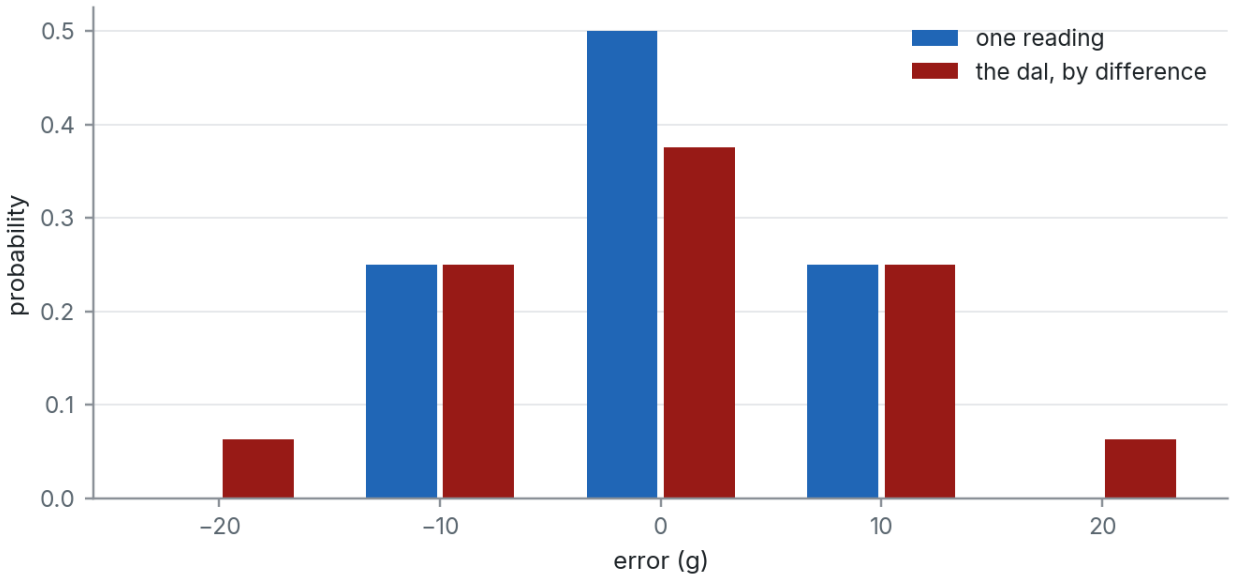
IN PLAIN TERMS. The expected value says where a chancy quantity sits on average. The variance says how far it strays from there.

Take each value's distance from the expected value and square it. Weight each square by that value's probability, and add them up.

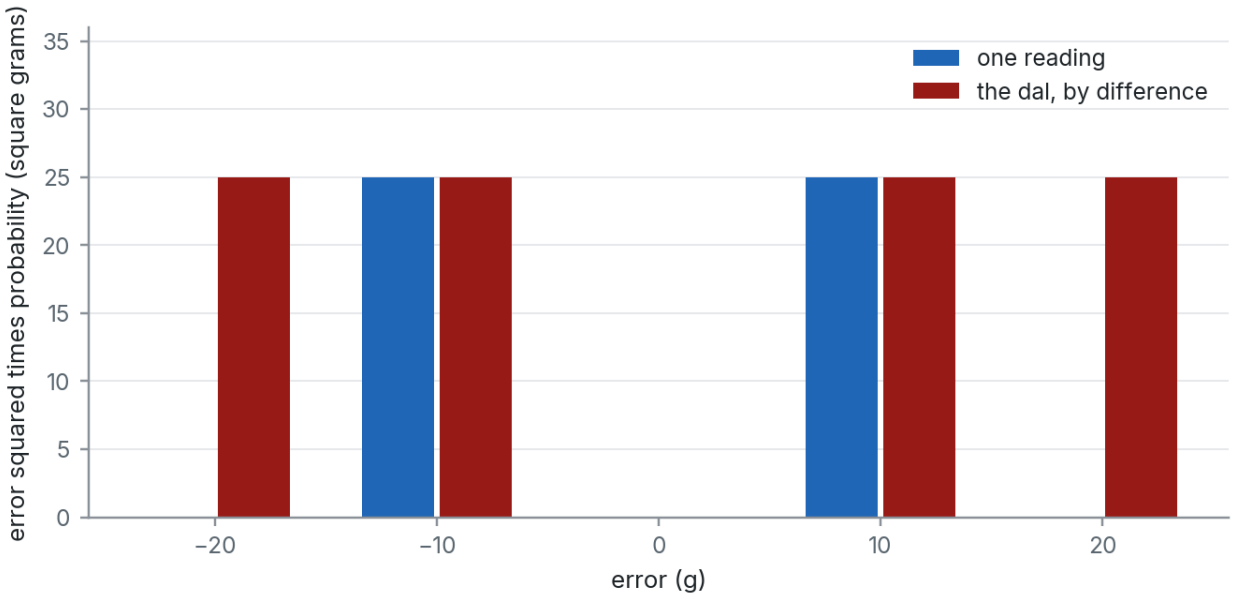
The answer is in squared units: square grams, if the quantity was in grams. Take its square root and you are back in grams.

Now add two uncertain amounts whose errors have nothing to do with each other. The spread of the total grows, but by less than the two spreads added.

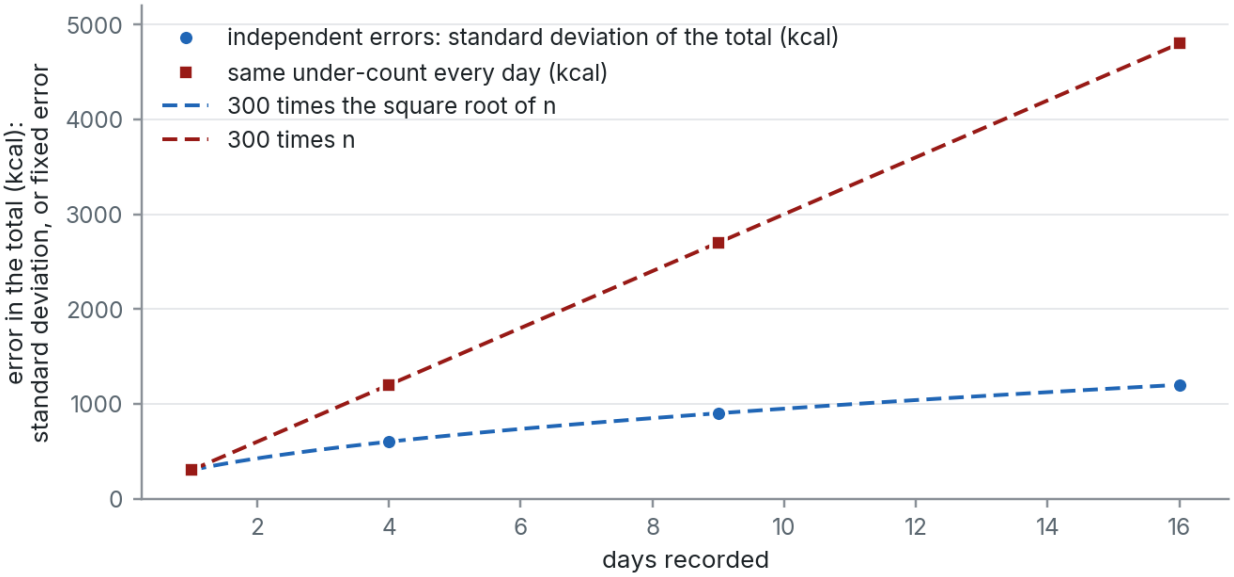
Subtract them instead, and the spread grows by exactly the same amount. Subtracting does not cancel uncertainty.



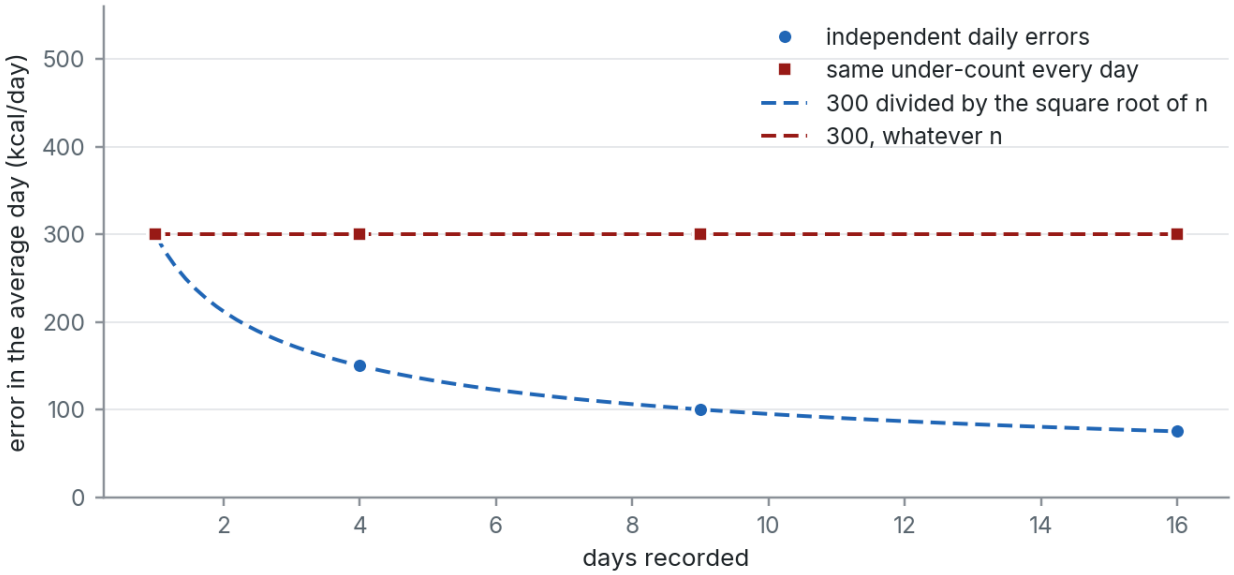
The error of one made-up reading (minus 10, 0 or 10 grams) and of the dal weighed by difference. The difference spreads from minus 20 to 20 grams: its variance is 100, the sum of the two readings' 50 and 50.



Each bar is one error's square times its probability: its share of the variance. One reading's two shares add to 50 square grams. The dal's four add to 100. Its errors of 10 grams either way carry the same 25 each as one reading's, and the new tails at minus 20 and 20 carry another 25 each, so subtracting doubles the variance.



The error in a total of n days, from made-up daily errors of 300 kcal. Independent errors grow as 300 times the square root of n, reaching 1200 at 16 days. The same under-count every day grows as 300 times n, reaching 4800.



Divide each total by the days to get the error in the average day. Independent errors shrink as 300 divided by the square root of n, to 75 kcal a day at 16 days: the square-root law. The same under-count every day stays at 300 kcal a day however long you record.

ILLUSTRATION. You weigh dal on a cheap kitchen scale, by difference. Weigh the empty katori. Put the dal in and weigh again. The dal's weight is the second reading minus the first.

Suppose each reading is off by minus 10, 0 or plus 10 grams, with probabilities 0.25, 0.5 and 0.25. Suppose too that the error of one reading tells you nothing about the error of the other, so the two are independent. Call the error of one reading X.

Find its expected value first: each value times its probability, added up.

```
-10 times 0.25 plus 0 times 0.5 plus 10 times 0.25 = 0
```

Now the variance. Each value's deviation from 0 is the value itself. Square it, weight it by its probability, and add.

```
100 times 0.25 plus 0 times 0.5 plus 100 times 0.25 = 50
```

Var(X) is 50 square grams. The standard deviation is the square root of 50, about 7.07 grams.

The dal's error is the second reading's error minus the first's. Work out its distribution by listing the pairs. Because the readings are independent, multiply to get each pair's probability, the rule from Book 0, D2. An error of minus 20 grams happens only when the second reading is 10 low and the first is 10 high.

```
0.25 times 0.25 = 0.0625
```

An error of minus 10 happens two ways: second 10 low and first right, or second right and first 10 high. Add the two ways.

```
0.25 times 0.5 plus 0.5 times 0.25 = 0.25
```

An error of 0 happens three ways: both low, both right, or both high.

```
0.25 times 0.25 plus 0.5 times 0.5 plus 0.25 times 0.25 = 0.375
```

The positive errors mirror the negative ones. Set the two distributions side by side.

error (g)	one reading	the dal, by difference
-20	0	0.0625
-10	0.25	0.25
0	0.5	0.375
10	0.25	0.25
20	0	0.0625

The dal's expected error is 0, by the same symmetry. Its variance, from the squared errors:

```
400 times 0.0625 plus 100 times 0.25 plus 0 times 0.375 plus 100 times 0.25 plus 400
times 0.0625 = 100
```

Now check the rule. The variance of a difference of independent errors is the sum of their variances.

```
50 plus 50 = 100
```

It agrees. The dal's standard deviation is the square root of 100, which is 10 grams. Each reading alone had about 7.07. You subtracted, and the error grew.

Adding the two standard deviations gives about 14.14 grams, and that is wrong. Variances add; standard deviations do not.

Now carry the rule from grams to energy. Hall and colleagues wrote a consensus statement on energy balance. It puts the uncertainty of expenditure measured by doubly labelled water at more than 100 kilocalories (kcal) a day. It gives the same figure as 420 kilojoules (kJ) a day.

Converting kcal to kJ multiplies every value by 4.184, the factor in Book 0, B4. So a spread in kJ is a spread in kcal times 4.184. Take 100 kcal a day as the floor of the standard deviation.

```
100 times 4.184 = 418.4
```

The paper rounds this to 420. Why does the standard deviation scale by 4.184 and the variance by 4.184 times 4.184? Derive it, naming each move. Call the error in kcal X, and the same error in kJ aX + b, with a = 4.184 and b = 0. Keep b in, because the same steps show what an added constant does.

First, the expected value of aX + b is a times E[X] plus b. The last section showed that.

Second, subtract it from aX + b. Substitute, then collect terms.

```
(aX + b) - (aE[X] + b) = aX - aE[X] = a(X - E[X])
```

The b has gone. Adding a constant moves every value and its expected value together.

Third, square both sides.

```
((aX + b) - E[aX + b])² = a² (X - E[X])²
```

Fourth, take the expected value of both sides. The a^2 is a fixed number, so it comes outside, by the rule for E[aX] from the last section. The left side is now the variance of aX + b. The right is a^2 times the variance of X.

```
Var(aX + b) = a² Var(X)
```

With a = 4.184, the variance in kJ is 4.184 times 4.184, about 17.5, times the variance in kcal. The standard deviation, its square root, is 4.184 times the standard deviation in kcal.

The energy imbalance of a day is energy intake (EI) minus energy expenditure (EE).

Take the expenditure error's standard deviation as 100 kcal a day, the floor from above. So suppose, as a made-up figure, that the intake error has a standard deviation of 300 kcal a day. Suppose too that the two errors are independent. The variance of the imbalance is the sum of the two variances.

```
100 times 100 plus 300 times 300 = 100000
```

The standard deviation of the imbalance is the square root of 100,000, about 316.2 kcal a day. It is at least that, because 100 was a floor.

Now add up days. Take two made-up cases, each with an intake error of 300 kcal on a single day. In the first, each day's error is independent of every other day's. In the second, the record undercounts by the same 300 kcal every day.

In the first case the variance of an n-day total is n times the variance of one day. So its standard deviation is 300 times the square root of n. Over 9 days the square root of 9 is 3.

```
300 times 3 = 900
```

In the second case nothing varies. The total error is 300 times n.

```
300 times 9 = 2700
```

days	independent errors: standard deviation of the total (kcal)	same under-count every day (kcal)
1	300	300
4	600	1200
9	900	2700
16	1200	4800

Divide by the days to get the error in the average day. In the first case it shrinks: 1,200 divided by 16 is 75 kcal a day, which is 300 divided by the square root of 16. That is the square-root law of Book 0, D6. In the second case it stays at 300 kcal a day however long you record.

Figures quoted in this illustration are from [9].

WHERE THIS PICTURE BREAKS. A real scale also has systematic error, the kind Book 0, D7 separates from random error. If the scale reads 5 grams heavy every time, that 5 grams sits in both readings and cancels in the difference. Weighing by difference removes an error shared by both readings and adds up the variances of the parts that are not shared.

The 300 kcal a day is made up, and so is the choice between the two cases.

The adding rule also needs the intake and expenditure errors to be independent. That is an argument you make, not a fact the numbers give you.

MUST KNOW POINTS FOR YOU.

- Subtracting one measured amount from another does not cancel their errors. Treat a small difference between two large measured amounts as the least trustworthy number on the page.
- Hall and colleagues say expenditure measured by doubly labelled water can be off by more than 100 kcal a day. Take measured intake minus that expenditure. If the two errors are independent, the gap you get is at least as uncertain. So do not accept one person's measured imbalance quoted as if it were known to within 50 kcal a day.
- Standard deviations do not add. Square each one, add the squares, and take the square root of the total.
- The adding rule, and the square-root growth over days, hold only for independent errors. An under-count repeated every day grows in step with the days, and averaging does not shrink it. Before you use the rule, make the independence argument Book 0, D2 asks for.
- Two measurements of the same person are rarely independent. A before-and-after change, or a predicted-minus-actual difference, can then spread much less, or more, than the adding rule says. A difference spreads less when its two parts rise and fall together, and more when one rises as the other falls. Do not call a reported spread wrong only because it fails the rule on paired figures.
- When a paper subtracts a baseline, ask whether the baseline was a fixed number or a measurement. Taking away a measured baseline adds the variance of the baseline's measurement error to the change, when that error is independent of the later reading's.
- Suppose a patient's food diary shows a deficit of a few hundred kcal a day and their weight has not moved. Do not treat the diary as the fixed truth and their body as the puzzle.
- Teach a trainee to convert a variance to a standard deviation before setting it beside a mean. A variance is in squared units, such as square grams.

EXERCISE 1 (interpretation). Here is a line from a methods section about the error in an energy imbalance. X is the error in measured intake, and Y the error in measured expenditure, both in kcal a day.

```
Var(X - Y) = Var(X) + Var(Y)
```

Say what each symbol denotes, and what unit each term carries. Say what condition the line needs, and why the minus sign on the left becomes a plus on the right.

EXERCISE 2 (teaching). A journalist asks why a study cannot simply measure whether a new snack changes a person's energy balance by 50 kcal a day. You have three minutes and no jargon. Give one quotable sentence.

PRACTICE. Eighteen problems on this technique, easiest first. Work them on paper before turning to the answers, which are in the appendix at the back under this section's name.

1. A random variable takes the values 2, 4 and 6 with probabilities 0.25, 0.5 and 0.25.

Work out its expected value, its variance and its standard deviation.

2. A random variable takes the value 0 with probability 0.8 and the value 10 with probability 0.2.

Work out its expected value, its variance and its standard deviation.

3. 1. A variance is 9. What is the standard deviation? 2. A standard deviation is 0.5. What is the variance? 3. A variance is 0.49. What is the standard deviation?

4. Var(X) = 4. Work out the variance and the standard deviation of 3X + 7, and of −2X + 5.

5. X and Y are independent. Var(X) = 9 and Var(Y) = 16.

Work out the variance and standard deviation of X + Y, and of X − Y. Then add the two standard deviations of X and Y, and compare.

6. Nine independent random variables each have standard deviation 2.

Work out the standard deviation of their sum, and the standard deviation of their mean.

7. A day's error has standard deviation 10. The errors on different days are independent. Over how many days does the standard deviation of the total reach 30?

8. Hall and colleagues give doubly labelled water "a precision of" about 5% (Hall 2012). They say this "translates to an uncertainty of energy expenditure of" more than 100 kcal a day, or 420 kJ a day.

1. At what daily expenditure is 5% exactly 100 kcal a day?
2. Take 100 kcal a day as a standard deviation. Convert it to kJ a day, using 4.184 kJ in a kcal, and compare with the paper's 420.
3. By what number is the variance multiplied in that conversion?

9. The error in a person's expenditure, measured by doubly labelled water, has a standard deviation of at least 100 kcal a day (Hall 2012). Suppose the error in their reported intake has a standard deviation of 250 kcal a day. That figure is made up. The two errors are independent.

Work out the standard deviation of the error in their energy imbalance, and say whether your answer is a ceiling or a floor.

10. Keep the imbalance error from the last problem, a standard deviation of about 269.3 kcal a day. Suppose the errors on different days are independent.

Work out the standard deviation of the error in the total imbalance over 7 days, and in the average daily imbalance over those 7 days.

11. Hall and colleagues write that "the combined error of assessing energy imbalance can easily reach 1000 kcal/d" (Hall 2012). They put the expenditure error at more than 100 kcal a day.

Read the 1,000 as a standard deviation, and the expenditure part as exactly 100. Suppose the intake and expenditure errors are independent. What standard deviation of the intake error would give the 1,000?

12. Here is a worked answer. Find the step that broke.

```
intake error: standard deviation 300 kcal a day
expenditure error: standard deviation 100 kcal a day, independent of the intake er-
ror
the imbalance is intake minus expenditure, so its error is 300 minus 100
standard deviation of the imbalance error = 200 kcal a day
```

13. Here is a worked answer. Find the step that broke.

```
Var(X) = 4
Var(3X) = 3 times Var(X) = 12
standard deviation of 3X = the square root of 12, about 3.46
```

14. Here is a worked answer. Find the step that broke.

```
in a made-up group, reported intake under-counts by 300 kcal on every day
over 30 days, the errors add up as the square root of the days
error in the 30-day total = 300 times the square root of 30, about 1643 kcal
```

15. Here is a worked answer. Find the step that broke.

```
the error of one reading on a scale has standard deviation 7 grams
a portion is weighed by difference: full bowl minus empty bowl
the bowl's weight is subtracted, so its error is removed
error of the portion = 7 grams
```

16. Thomas and colleagues pooled seven weight-loss studies. They report that participants "lost 20.1±11.3 lbs, 7.4±12.6 lb less than the 27.6±16.0 lbs predicted by the 3500 kcal rule" (Thomas 2013), the static rule of Book 1, section 7. Their methods paragraph writes figures in the form mean ± standard deviation.

A reader says: "The difference is predicted minus actual, so its standard deviation should be the square root of 11.3 squared plus 16.0 squared. That is about 19.6, not 12.6, so the 12.6 is a typo."

Decide what to compute, compute it, and say what your answer does not establish.

17. A calorie-counting app's advert says this.

> *Our entries are accurate to within 50 kcal per item. So your daily log of 12 items is accurate to within 50 kcal.*

The figures are the advert's own and are made up for this problem. Decide what to compute, compute it, and say what your answer does not establish.

18. Thomas and colleagues report three figures for the same 103 adults (Thomas 2013). Baseline energy intake was 2876±484 kcal a day. The prescribed target intake was 1409±569 kcal a day. And they report "an average deficit of 1439±784 kcal/d". The figures after ± are standard deviations.

A reviewer writes: "The deficit is baseline intake minus target intake. If those are independent, its standard deviation must be the square root of 484 squared plus 569 squared. That is about 747, so the reported 784 is too large."

Decide what to compute, compute it, and say what your answer does not establish.

16 · Following a derivation without skipping

DEFINITION. A derivation is a chain of equations in which each line follows from lines above it by one move that can be named. The moves this book uses are these.

- Substitute: replace a symbol by something an earlier line states is equal to it.
- Rearrange: do the same thing to both sides, by the four operations of Book 0, C2.
- Divide by a time interval and shrink it: turn a change over an interval into a rate at an instant, the derivative of section 2.
- Differentiate both sides, or integrate both sides over an interval, as section 3 and section 5 do.
- Assume: bring in a new premise.

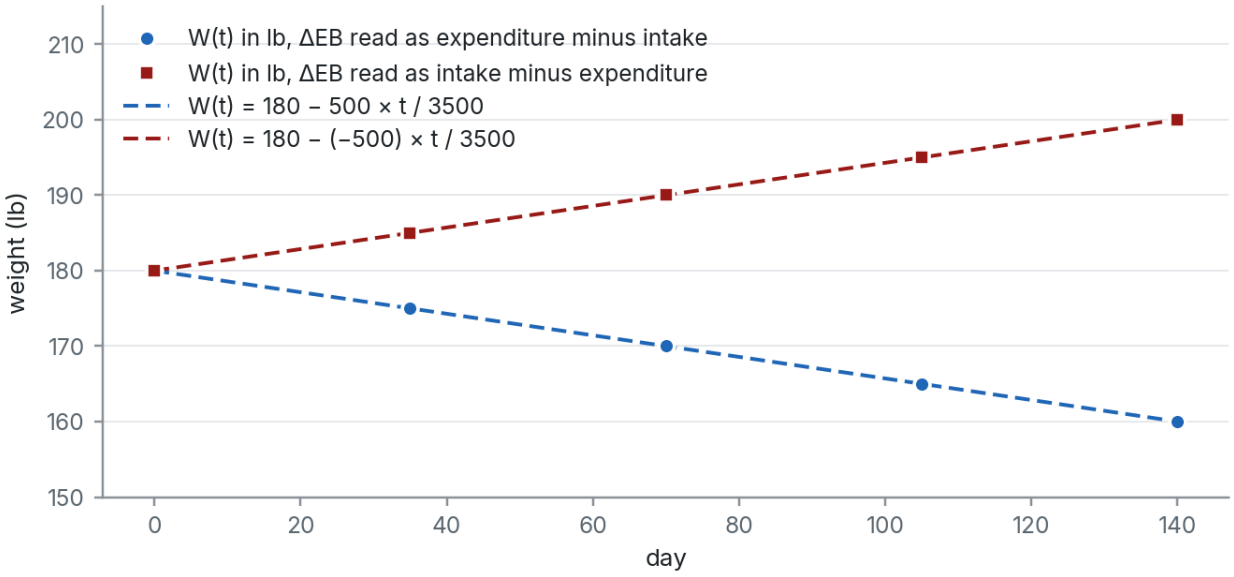
Putting a value in place of a symbol is a substitution. Doing the arithmetic on one side, or collecting like terms, is part of rearranging. An assumed line carries nothing from above.

A where-clause fixes each symbol's meaning, its unit and its sign. The same printed equation can say opposite things under two readings of its where-clause. [126, 127, 128, 37, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138]

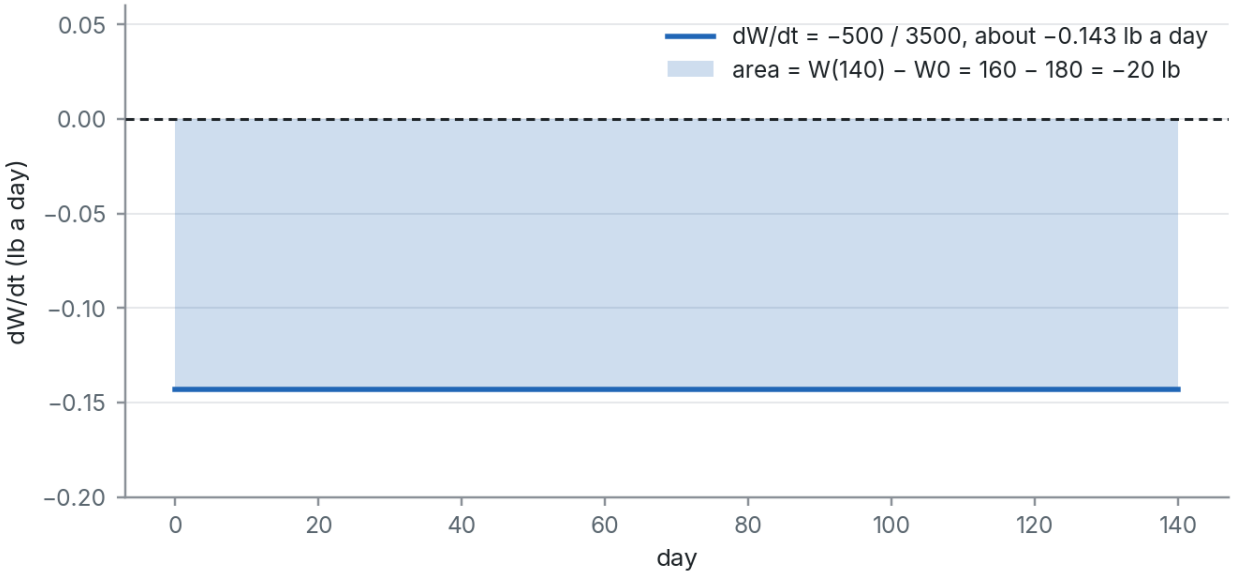
IN PLAIN TERMS. A derivation is a staircase. Each step down is one small move you could make yourself.

Some steps are not moves at all. The author brings in something new: "suppose intake minus expenditure stays fixed". That line is a premise. Everything below it depends on it, however careful the steps that follow.

So read a derivation with a pencil. Beside every line, write the move that got you there.



The printed equation W(t) = W0 − ΔEB × t / 3500 under two readings of ΔEB, for a made-up start of 180 pounds and a deficit of 500 kcal a day. Read as expenditure minus intake, weight falls to 160 by day 140; read the other way, the same equation makes it rise to 200. The weight axis does not start at zero.



Line 5, with ΔEB as line 4 defines it (intake minus expenditure, so −500), makes the rate of change of weight a constant: minus 500 divided by 3,500 pounds a day, for a deficit of 500 kcal a day. Line 6 integrates it from day 0 to day 140. The shaded area, 140 times that rate, is minus 20 pounds, the fall from 180 to 160 in the table. The area is a rectangle only because line 4 assumed the gap never changes.

ILLUSTRATION. Start with the rule Book 1 called the static rule, as Thomas and colleagues print it (Thomas 2013). They write W(t) for the expected weight, in pounds, on day t, and W0, said “W nought”, for the weight at the start. They write ΔEB for “the change in energy balance in kcal/d”.

```
W(t) = W0 - ΔEB × t / 3500
```

Rebuild it from the identity, one move per line.

Line 1 is the identity as a rate.

```
dES/dt = EI - EE (line 1: the identity, taken as given)
```

Line 2 brings in something new. The rule assumes every 3,500 kcal of stored energy gained or lost is one pound of body weight.

```
dES/dt = 3500 × dW/dt (line 2: assumed, 3,500 kcal in every pound, written as rates:
divide the change by the interval and shrink it)
```

```
3500 × dW/dt = EI - EE (line 3: substitute)
```

Line 4 is a second assumption. Intake minus expenditure stays the same number on every day. Call that number ΔEB.

```
3500 × dW/dt = ΔEB (line 4: assumed, the gap never changes)
```

```
dW/dt = ΔEB / 3500 (line 5: rearrange)
```

```
W(t) - W0 = ΔEB × t / 3500 (line 6: integrate both sides from 0 to t)
```

```
W(t) = W0 + ΔEB × t / 3500 (line 7: rearrange)
```

Now set your line 7 beside the printed equation. Yours has a plus where theirs has a minus. The difference sits in the where-clause. Thomas and colleagues define ΔEB as “the difference between the rate of energy intake and the rate of energy expenditure”. That phrase does not say which is taken from which.

The printed equation predicts a fall only if ΔEB is expenditure minus intake: the deficit, written as a positive number.

Take a made-up case: start at 180 pounds, with a deficit of 500 kcal a day.

day	W(t) in lb, ΔEB read as expenditure minus intake	W(t) in lb, ΔEB read as intake minus expenditure
0	180	180
35	175	185
70	170	190
105	165	195
140	160	200

Line 1 is physics. Lines 3, 5, 6 and 7 are safe moves. Lines 2 and 4 are assumptions. Hall and colleagues reject line 4. Expenditure falls as weight falls, so the gap in line 4 shrinks.

Polidori and colleagues work out the change in each person’s energy intake from their weights, with their Equation 1 (Polidori 2016). Here W is in kilograms again, as in the rest of the book. The equation’s first two terms, in this book’s symbols (W for their BW), are:

```
ΔEI = ρ × dW/dt + ε × (W - W0)
```

Its other terms, for activity and for glucose lost in the urine, come back in problem 9.

The Greek letter rho, ρ, is "the effective energy density associated with the body weight change", in kcal per kg. The Greek letter epsilon, ε, sets "how energy expenditure depends on body weight", in kcal a day per kg.

Line 1 is the model of section 7.

```
ρ × dW/dt = EI - EE(W) (line 1: the model, taken as given)
```

Line 2 assumes expenditure is a straight line in weight near the start. It is its starting value, EE0, plus ε times the change in weight.

```
EE(W) = EE0 + ε × (W - W0) (line 2: assumed, a straight line near the start)
```

Line 3 writes intake as its starting value, EI0, plus a change, ΔEI. This defines ΔEI. It assumes nothing.

```
EI = EI0 + ΔEI (line 3: definition)
```

```
ρ × dW/dt = EI0 + ΔEI - EE0 - ε × (W - W0) (line 4: substitute)
```

Line 5 needs a premise. At the start, weight was steady, so its rate was zero. That is the equilibrium of section 8: intake equalled expenditure, and EI0 minus EE0 is zero.

```
ρ × dW/dt = ΔEI - ε × (W - W0) (line 5: assumed, steady at the start)
```

```
ΔEI = ρ × dW/dt + ε × (W - W0) (line 6: rearrange)
```

The equation that turns weights into intake is the balance of section 7 plus premises. Each premise is a place a reader can push.

Figures quoted in this illustration are from [9].

WHERE THIS PICTURE BREAKS. The two straight lines in the table hold only while lines 2 and 4 hold.

The rebuild also takes ρ and ε as given. The paper does not print this route, and its ρ and ε are built from further parts (its Equations 2 and 3). The rebuild shows the equation's shape, not how the authors reached it.

MUST KNOW POINTS FOR YOU.

- Beside every line of a derivation, write the move that produced it. If you cannot name one, the line is a skipped step or a premise.

- A result derived from the first law is not thereby proved. When someone says an equation "comes from physics", ask which lines are physics and which are assumptions.
- Test a printed equation on a case whose answer you know. A deficit must lower weight. If your reading makes it raise weight, your reading is wrong.
- Check the unit of every term on every line.
- Following a derivation tells you whether each line follows. It cannot tell you whether an assumed line is true; that needs evidence, as the passive fall in expenditure (section 3) is for the 3500 kcal rule.
- When you teach a derivation, have the trainee mark assumed lines differently from derived ones, before they say anything about the result.
- When a press line says a diet "must" produce a steady loss "because of physics", the accurate reply is short. The physics gives one line. The steady loss needs a second line, that intake minus expenditure stays fixed, and it does not.

EXERCISE 1 (teaching). A first-year resident asks where the rule "3,500 kcal a pound" comes from and why the consensus panel dropped it. You have ten minutes and a whiteboard. Plan what you will write, line by line.

EXERCISE 2 (critique). A trainee's journal-club slide says: "Polidori and colleagues derive Equation 1 from the first law of thermodynamics, so the intake changes it gives are exact."

Find the specific flaw.

PRACTICE. Fourteen problems on this technique, easiest first. Work them on paper before turning to the answers, which are in the appendix at the back under this section's name.

1. Write the move that produced each line from the line above it.

```
4x + 6 = 26
4x = 20
x = 5
```

2. Write the move that produced the third line.

```
y = 3x
x = 5
y = 15
```

3. The author went from the first line to the last in one step. Fill in the skipped lines, one move each, and name each move.

```
5x - 4 = 2x + 11
x = 5
```

4. Name the move from the first line to the second, and the rule it uses.

```
y = 4t + 3
dy/dt = 4
```

5. A quantity W is 80 at t = 0 and its rate of change is dW/dt = −0.2 at every t. Write the lines that give W at t = 10, with the move beside each.

6. Find the line that does not follow from the one above it, and say what it should be.

```
4x + 8 = 20
x + 8 = 5
x = -3
```

7. Use the equation W(t) = W0 − ΔEB × t / 3500, with W in pounds and ΔEB in kcal a day (Thomas 2013). Take the reading that makes a deficit lower weight. A person starts at 200 pounds with a deficit of 700 kcal a day, figures made up for this problem.

Work out W(28), and say which reading of ΔEB you used.

8. Chow and Hall describe one step in words (Chow and Hall 2008). They “divide Equation 1 by some interval of time and take the limit of infinitesimal change” (infinitesimal: shrinking towards zero, the limit of section 2).

Apply that step to Book 1’s identity over an interval of Δt days. Write ES for body energy stores in kcal.

```
ΔES = (energy taken in over the interval) - (energy spent over the interval)
```

Write the lines that take it to Hall’s rate form, naming each move, and give the unit of each side at each line.

9. In Polidori and colleagues’ trial the drug makes the body lose extra glucose in the urine, which the paper calls UGE (urinary glucose excretion). Write it U, energy lost a day, zero before the drug starts. The balance becomes:

```
ρ × dW/dt = EI - EE(W) - U
```

Using the same premises as the section’s rebuild, derive an equation for ΔEI, naming each move. Then compare it with the paper’s Equation 1 with Δδ = 0 (Polidori 2016).

10. Here is a worked answer, with made-up intake and expenditure. Find the step that broke.

```
3500 × dW/dt = EI - EE
dW/dt = EI - EE / 3500
with EI = 2000 and EE = 2500, dW/dt = 2000 - 0.714 = 1999.3 pounds a day
```

11. Here is a worked answer. Find the step that broke.

```
ρ × dW/dt = EI - EE(W)
EE(W) = EE0 + ε × (W - W0), and EI = EI0 + ΔEI
ρ × dW/dt = EI0 + ΔEI - EE0 - ε × (W - W0)
so ΔEI = ρ × dW/dt + ε × (W - W0)
```

12. Here is a worked answer. Find the step that broke.

```
a made-up rate of change of weight: dW/dt = -0.1 plus 0.001 t, in kg a day, t in
days
integrate both sides from 0 to 60: the change is the rate times the time
at t = 60 the rate is -0.1 plus 0.06 = -0.04
change = -0.04 times 60 = -2.4 kg
```

13. Thomas and colleagues report a mean deficit of 1439 kcal a day over a mean of 64.8 days (Thomas 2013). The mean loss the 3500 kcal rule predicted was 27.6 pounds.

A colleague says: “1439 times 64.8 divided by 3500 is 26.6, not 27.6, so their prediction has an arithmetic slip.”

Decide what to compute, compute it, and say what your answer does not establish.

14. A health column says: “Physics proves that cutting 500 kcal a day loses a pound a week, for as long as you keep it up.”

Decide what to compute, compute it, and say what your answer does not establish.

17 · Four kinds of equation, and the notebook entry

DEFINITION. Four kinds of equation are worth telling apart when you read, and each is checked a different way. Papers hold other kinds too; later books meet them.

An identity holds for every value of its symbols because a conservation law, or counting, requires it.

A definition brings in a new quantity as a combination of others. It cannot be false. A definition rearranged is still a definition.

A model assumption is a claim, adopted by the author, about how a body or a population behaves. It could be false.

A fitted equation has a form chosen by its author and constants estimated from one set of measurements.

One printed equation can mix kinds: a model assumption with a fitted constant, or a definition rearranged and then used to estimate a quantity from inputs that are themselves fitted or adopted. Each part is then checked by its own kind. A model's output inherits the kinds of the model that produced it, and a fitted value can then be adopted by agreement. Record both.

A notebook entry records one equation so that it can be used and checked later. It holds six things.

1. The equation as printed.
2. Each symbol in words, with its unit.
3. The whole equation said as one sentence.
4. Its kind.
5. What it assumes, or whom it was fitted on.
6. The paper, with the equation's number or page. [139, 140, 141, 142, 143, 144, 145, 146, 97, 147, 148, 149, 135, 150, 151, 54, 152, 153, 154, 136, 155, 156, 157]

IN PLAIN TERMS. Equations in papers look alike: letters, an equals sign, more letters. They are not alike. Some are true whatever happens. Some are true because someone decided what a word means. Some are guesses about how bodies behave. Some are lines drawn through somebody else's measurements.

Which kind you are looking at decides what you check.

So keep a notebook.

ILLUSTRATION. Start with an equation that went wrong for Indian adults, and find out why by sorting it.

FAO/WHO/UNU 2004, the report of a consultation, prints equations for basal metabolic rate (BMR). BMR is the energy a body spends lying awake and at rest, after an overnight fast. The equations come from Schofield, and the consultation "decided to retain" them.

```
BMR = 15.057 × weight + 692.2
```

Now sort it. It is not an identity: nothing forces a body's BMR to be this number. It is a fitted equation. Its form, a straight line in weight, was chosen.

So ask the fitted equation's question: whom was it fitted on? The Indian Council of Medical Research and National Institute of Nutrition (ICMR-NIN) answers it for you. The FAO/WHO/UNU equations "can overestimate the BMR by 10 -12%, because the FAO/WHO/UNU data set included young and muscular subjects."

ICMR-NIN's complaint is not with the line but with whom it was drawn through.

Now write the entry.

- Each symbol with its unit: BMR, basal metabolic rate, in kcal a day. Weight in kg. 15.057, in kcal a day for each kg. 692.2, in kcal a day.
- In words: for men aged 18 to 30, BMR is about 15 kcal a day for each kilogram, plus about 692 kcal a day.
- Kind: fitted.
- Assumes, or fitted on: Schofield's measured men, 2,879 in this band.
- Source: FAO/WHO/UNU 2004, chapter 5, Table 5.2.

A second entry shows an equation of two kinds at once. Polidori and colleagues write their Equation 4 for how intake answers weight loss (Polidori 2016).

```
ΔEI(t) = -kP × ΔBW(t)
```

- Symbols: the change in intake from the start, ΔEI(t), is in kcal a day. The change in body weight (BW) from the start, ΔBW(t), is in kg. The constant kP, above zero, is "the feedback strength", in kcal a day per kg. The minus sign makes intake rise when weight falls.
- In words: for every kilogram lost, intake rises by kP kcal a day.
- Kind: a model assumption with one fitted constant. The straight-line form is the assumption.
- Assumes: intake answers weight in proportion.
- Fitted on: the group means of a 52-week trial of canagliflozin in people with type 2 diabetes.
- Source: Polidori and colleagues 2016, Methods, Equation 4.

Figures quoted in this illustration are from [16, 98, 10].

WHERE THIS PICTURE BREAKS. ICMR-NIN's finding is about groups of Indian adults. It is not a correction to apply to a single patient's measured BMR.

Sorting is not always this clean. Take the Atwater factors of section 10: 4, 9 and 4 kcal a gram. They are fitted: averages of measurements, rounded. They were then adopted by agreement. Record both kinds when an equation has both.

MUST KNOW POINTS FOR YOU.

- Say an equation's kind before you use it, because the check differs. For a definition, read the where-clause. For a model assumption, ask for the evidence. For a fitted equation, ask whom it was fitted on. For an identity, check that every term is counted across the same boundary and over the same interval.
- An identity cannot tell you what to change. When someone offers the identity as a reason for an instruction, ask which model assumption they have added to it.
- ICMR-NIN's 2020 note finds that the FAO/WHO/UNU equations for BMR can run 10 to 12% high for Indians. For an Indian patient, use instead the requirements in the note's Tables 1a and 1b. They build in the reduction.
- An energy requirement is BMR times the physical activity level (PAL). ICMR-NIN's 2020 note cut the sedentary figure from 1.53 to 1.40. At the same BMR, that alone lowers the estimate by about 8.5%. Quote such a table with the year of its inputs.
- A fitted equation holds for people like those it was fitted on, over the range it was fitted. Outside them, use an equation fitted on people like yours, or a stated correction such as ICMR-NIN's, and say which you used.
- Polidori's about 100 kcal a day per kg was fitted to group means. Do not tell a patient their appetite will rise by exactly that much.
- A notebook entry without its kind and its assumptions is not finished. When you copy an equation from a typeset image, have a second reader check it symbol by symbol.
- Teach a trainee that a definition cannot be confirmed by data.

EXERCISE 1 (build). Start the notebook this book has been building towards: ten equations from papers you have read, each with a full six-part entry.

Here are ten candidates from papers this book has quoted. Choose any ten, from here or from your own reading. For each, write the six parts.

1. The energy balance equation, ES = EI − EO (Hall and colleagues 2012, Question 1).
2. PAL = TEE / BMR (FAO/WHO/UNU 2004, glossary and chapter 5).
3. TEE = BMR × PAL (FAO/WHO/UNU 2004, chapter 5; ICMR-NIN 2020).
4. One row of Table 5.2, such as BMR = 14.818 × weight + 486.6 for women aged 18 to 30 (FAO/WHO/UNU 2004). Table 5.2's "No." column gives how many were measured.
5. Energy in kcal = 4 × protein + 9 × fat + 4 × carbohydrate, each in grams (FAO Food and Nutrition Paper 77, §3.5.1).
6. W(t) = W0 − ΔEB × t / 3500 (Thomas and colleagues 2013, Methods).

7. $\Delta EIi = \rho \times dBWi/dt + \varepsilon \times (BWi - BW0) + (\Delta\delta / (1 - \beta)) \times BW0 + UGE$ (Polidori and colleagues 2016, Equation 1).
8. $\Delta EI(t) = -kP \times \Delta BW(t)$ (Polidori and colleagues 2016, Equation 4).
9. $\Delta EI(t) = -kP \times \Delta BW(t) - kI \times \int$ from 0 to t of $\Delta BW(\tau)\, d\tau$ (Polidori and colleagues 2016, Equation 5). τ here is the running time inside the integral, not the time constant of section 8.
10. The rule of thumb (Hall and colleagues 2012, Question 3). Eventual weight change in pounds = the permanent change in intake in kcal a day, divided by 10.

EXERCISE 2 (teaching). A state health secretary asks why the national energy requirement for sedentary adults fell in 2020, when nobody's body changed. One page, answer first, no methods section.

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Appendix · Worked answers

1 · Reading the symbols in a methods section

EXERCISE 1. Go symbol by symbol.

symbol	say it	what it denotes	unit
ΔEI(t)	delta E I of t	the change in energy intake from baseline, at time t	kcal a day
t	t	time	days
ΔBW(t)	delta B W of t	the change in body weight from baseline, at time t	kg
kP	k sub P	the strength of the feedback, a number greater than zero	kcal a day per kg

The (t) is function notation from Book 0, C4. ΔEI(t) is the change in intake at time t. It is not ΔEI times t.

The unit of kP comes from the unit check. The left side is in kcal a day. ΔBW(t) is in kg. So kP must be in kcal a day per kg.

The minus sign reverses the direction. Weight lost makes ΔBW(t) negative. Minus a positive kP times a negative number is positive, so intake rises above baseline when weight falls.

The abstract's figure is kP, and its unit is exactly the one the check predicts: about 100 kcal a day per kg of lost weight. That is the paper's rounded figure. The Results give the fitted value, 95.

```
a loss of 1 kg: ΔBW = -1
ΔEI = minus 100 times -1 = 100
```

About 100 kcal a day more eating for each kilogram lost, on this model and for the group average.

1. Put i = 1, 2, 3 and 4 into 3i in turn, then add.

```
3 times 1 plus 3 times 2 plus 3 times 3 plus 3 times 4
3 plus 6 plus 9 plus 12 = 30
```

The sum is 30.

2. Each term is 2 times its place in the list: the first is 2 times 1, the fifth is 2 times 5. So the counter runs from 1 to 5 and each term is 2i.

```
Σ[i = 1 to 5] 2i
```

Check by writing it out again.

```
2 times 1 plus 2 times 2 plus 2 times 3 plus 2 times 4 plus 2 times 5 = 30
2 plus 4 plus 6 plus 8 plus 10 = 30
```

3. The first adds all four.

```
4 plus 7 plus 1 plus 8 = 20
```

The second starts at i = 2 and stops at i = 3, so it adds only x2 and x3.

```
7 plus 1 = 8
```

The third multiplies each value by 5 before adding.

```
5 times 4 plus 5 times 7 plus 5 times 1 plus 5 times 8
20 plus 35 plus 5 plus 40 = 100
```

That is 5 times the first sum, 5 times 20. A number that multiplies every term can be taken outside the sum.

4. A term added to B must carry B's unit. So C × D is in kilocalories a day, and A, which equals the sum, is in kilocalories a day too.

C times kilograms gives kilocalories a day. So C is kilocalories a day divided by kilograms.

```
C = (kcal a day) divided by (kg)
= kcal a day per kg
```

A is in kcal a day, and C is in kcal a day per kg.

5. Rearrange PAL = TEE/BMR for TEE by multiplying both sides by BMR (Book 0, C2).

```
TEE = PAL times BMR
1.75 times 7.10 = 12.425
```

12.425 MJ a day, which the report prints cut to 12.42.

Now the unit of PAL. It is TEE divided by BMR.

```
(MJ a day) divided by (MJ a day)
```

The two units cancel, so PAL has no unit. It says how many times the basal rate the day's total is. A multiplier that turns one energy a day into another has to be a bare number.

6. ρF is "rho sub F": the energy density of the body fat mass change, in megajoules per kilogram. The F labels which energy density, the one for fat. ΔF is "delta F": the change in body fat, in kilograms.

```
39.5 times -3 = -118.5
(MJ per kg) times (kg) = MJ
```

Minus 118.5 megajoules. The kilograms cancel and leave energy.

The minus sign comes from ΔF. Fat fell by 3 kg, so the energy held in that fat fell by 118.5 megajoules.

7. N is a count of weighings in one interval, with no unit. T is the number of days between two weighings, so it is in days. t is the length of the interval, in days.

The asterisk is a times sign, as on a keyboard.

```
t = (N minus 1) times T
(2 minus 1) times 52 = 52
```

Each interval is 52 days long. Two weighings mark out one gap. Here t is a length of time, not a moment.

With three weighings there are two gaps.

```
(3 minus 1) times 52 = 104
```

104 days.

8. The third step broke. It puts 90 grams a day into a sum of kilocalories a day.

Every term in Equation 1 must carry the unit of the left side, kcal a day. A figure in grams of glucose a day is not yet an energy. It has to be an energy in kilocalories a day first.

So 340 is not a number of anything. It adds two quantities of different kinds. The right answer needs UGE as an energy, which the paper gives in its Table 1.

Why the wrong answer looks reasonable. The 90 is the paper's own figure, and the addition is correct arithmetic. The unit was left off the third line, and a number with no unit beside it looks like any other number.

9. The third step broke. BW0 is not BW times 0. The 0 is a subscript, part of the name. BW0 is body weight at baseline, the weight at the start.

The rule in the second line is right for most symbols side by side. A subscript is one of the exceptions.

The term is ε times the change in weight since baseline. For that you need BW0 as a weight. Say the baseline was 84 kg, a made-up figure.

```
80 minus 84 = -4
20 times -4 = -80
```

Minus 80, not 1600. The broken reading turned a change of 4 kg into a weight of 80 kg.

Why the wrong answer looks reasonable. It applies a rule you were right to learn, and every later line follows from the third. Written on the line, a subscript 0 looks like a number.

10. What to check: whether ES in the equation is the store, which the colleague assumes. Use the unit check, and then the paper's words.

The paper says of the terms: "All of these terms are expressed as energy per unit of time."

```
EI minus E0: (kcal a day) minus (kcal a day) = kcal a day
the stores: kcal, with no time in the unit
```

The two sides cannot match if ES is the store. And the paper says so in words: "ES is the rate of change in the body's macronutrient stores."

So the equation says nothing directly about the 130,000 kcal. It says how fast the stores change on a given day. To reach a store from rates you would have to add up the daily differences over the whole of a life, from a starting store.

What this does not establish. It does not say the colleague's conclusion is false. Stores do reflect intake minus expenditure added up over time. But it says the equation, read as written, is not the step that gets you there. It also says nothing about how the stores were built, or when.

11. What to check: the arithmetic, then the units, then whether the two ρ symbols are the same quantity.

The arithmetic first. A megajoule is 1,000 kilojoules, and 1 kcal is 4.184 kJ.

```
39.5 times 1,000 = 39,500
39,500 divided by 4.184 = 9,440.7
```

About 9,441 kcal. The sum is right.

The units. Both symbols are energy per kilogram, so the unit check passes. That is the trap: it cannot tell them apart.

The names. Hall's symbol is ρF, rho sub F, "the energy density of the body fat mass change". The subscript F says it is for fat alone. Polidori's ρ has no subscript and is "the effective energy density associated with the body weight change", all of it. Hall (2008) also reports Forbes's point that "lean body mass is lost in concert with body fat during weight loss". So a kilogram of weight change is not a kilogram of fat, and the two ρ symbols are different quantities.

What this does not establish. It does not give you Polidori's value of ρ. That comes from their Equation 2, which needs quantities this section has not read. It shows only that 39.5 belongs to fat, and that the note borrowed it by matching a letter.

2 · The derivative: the rate at an instant, computed

EXERCISE 1. dBWi/dt, said "dee B W sub i by dee t", is the rate of change of body weight over interval i. Weight is in kilograms and time in days, so it is in kilograms a day. A negative value means weight was falling over that interval.

Why not a day apart. Weight moves from day to day with body water, which Hall and colleagues (2012) call "variable", and, most likely, with food in the gut. In this section's made-up weighings, one day's swing was larger than the whole trend. A rate from two weighings a day apart could mostly measure the swings. Over 52 days the trend adds up and a day's swing does not, so the swing is small beside the trend.

What it gives up. The rate is an average over each interval. It cannot show a change that happens inside the interval, such as weight falling fast for two weeks and then stopping. In this book's terms it is the slope of a chord across the interval, not a tangent at a point.

1. The change in f, divided by the change in the input.

```
33.5 minus 30 = 3.5
4.5 minus 4 = 0.5
3.5 divided by 0.5 = 7
```

The average rate is 7.

2. f(3) = 9. For each h, work out f(3 + h), subtract 9, and divide by h.

```
h = 1: 4 times 4 = 16, 16 minus 9 = 7, 7 divided by 1 = 7
h = 0.1: 3.1 times 3.1 = 9.61, 9.61 minus 9 = 0.61, 0.61 divided by 0.1 = 6.1
h = 0.01: 3.01 times 3.01 = 9.0601, 9.0601 minus 9 = 0.0601, 0.0601 divided by 0.01
= 6.01
h = -0.1: 2.9 times 2.9 = 8.41, 8.41 minus 9 = -0.59, -0.59 divided by -0.1 = 5.9
h = -0.01: 2.99 times 2.99 = 8.9401, 8.9401 minus 9 = -0.0599, -0.0599 divided by
-0.01 = 5.99
```

From above: 7, 6.1, 6.01. From below: 5.9, 5.99. Both close in on 6.

The derivative at t = 3 is very likely 6. The algebra in problem 9 gives exactly 6: f′(3) = 6.

3. The first is “dee W by dee t”: the rate of change of W with respect to t. Kilograms divided by days, so kilograms per day.

The second is “f prime of 10 equals minus 3”. The rate of change of f with respect to body weight, at a weight of 10 kg, is minus 3. Its unit is kcal a day divided by kg: kcal a day per kg. The minus sign says f falls as weight rises, at that weight.

The second rate is not against time. A derivative can be taken with respect to any input, and its unit is always the output’s unit divided by the input’s.

4. Multiply out the bracket first.

```
(1 + h) times (1 + h) = 1 times (1 + h) + h times (1 + h)
= 1 + h + h + h^2
= 1 + 2h + h^2
```

Then multiply by 3, and subtract f(1), which is 3.

```
f(1 + h) = 3 + 6h + 3 h^2
f(1 + h) minus f(1) = 6h + 3 h^2
(6h + 3 h^2) divided by h = 6 + 3h
```

As h shrinks, 3h shrinks to nothing and 6 is left. The derivative at t = 1 is 6.

5. Read the sign.

At 0, positive, so rising. At 2, zero, so neither rising nor falling at that instant. At 5, negative, so falling. At 7, negative, so falling, and faster than at 5, because 12 is larger than 0.3.

The sign gives the direction and the size gives the speed. Neither says where the function is, only which way it is heading.

6. Change divided by time, keeping the order.

```
92.0 minus 94.6 = -2.6
-2.6 divided by 52 = -0.05
```

Minus 0.05 kg a day. Weight fell.

A week is 7 days.

```
-0.05 times 7 = -0.35
```

Minus 0.35 kg a week. Both are averages over the 52 days, not the rate on any one day.

7. The rate of change of stores is intake minus expenditure.

```
7.80 minus 8.26 = -0.46
```

Minus 0.46 MJ a day. Now kilocalories. A megajoule is 1,000 kilojoules.

```
0.46 times 1,000 = 460
460 divided by 4.184 = 109.94
```

About minus 110 kcal a day. The minus sign says her stores were falling that day.

8. New value is old value plus rate times time, the move from Book 0, C8. The rate is negative.

```
-0.05 times 10 = -0.5
92.0 plus -0.5 = 91.5
```

About 91.5 kg in 10 days.

```
-0.05 times 300 = -15
92.0 plus -15 = 77
```

About 77 kg in 300 days.

Trust the 300-day figure much less. The estimate holds the rate fixed. Over 10 days that is close to true. Over 300 days it is not, because the rate itself changes.

Hall and colleagues (2012) state that "weight change will slow over time due to passive compensatory changes in energy expenditure". As weight falls, expenditure falls with it. A slower rate means less than 15 kg lost. The rate at an instant is a good guide only for a short step ahead.

9. The second step broke. (3 + h) times (3 + h) is not $9 + h^2$. Each term in the first bracket multiplies each term in the second.

```
(3 + h) times (3 + h) = 3 times (3 + h) + h times (3 + h)
= 9 + 3h + 3h + h^2
= 9 + 6h + h^2
```

Then the rest follows.

```
f(3 + h) minus f(3) = 6h + h^2
(6h + h^2) divided by h = 6 + h
```

As h shrinks, 6 is left. The derivative at t = 3 is 6, not 0.

A quick check would have caught it. Take h = 0.1 in the table way: 3.1 times 3.1 is 9.61, so the average is 6.1, nowhere near 0.1.

Why the wrong answer looks reasonable. Squaring each part of a sum looks natural, and every line after the second is correct algebra. The last line even sounds like a real finding.

10. Two steps broke, the third and the fourth.

The third takes a rule that holds for a smooth curve and applies it to two weighings. Weight on a scale swings from day to day with water and, most likely, gut contents. A one-day interval sits inside those swings, so the rate it gives can be mostly swing.

The fourth turns a change in weight into a change in fat. Hall and colleagues (2012) warn that "weight change may not directly represent energy imbalances, particularly over the short term". Check the size.

```
0.6 times 39.5 = 23.7
```

Adding 0.6 kg of fat would take 23.7 MJ of stored energy in one day. That is nearly three days of the 8.26 MJ a day the FAO/WHO/UNU report gives for an average lightly active woman of 30 to 50.

What to do instead: take the rate from averages of several weighings, over weeks.

Why the wrong answer looks reasonable. The subtraction and the unit are right, and "a shorter interval is closer" is a rule this section taught. It was used where it does not hold.

11. What to compute: the average rate over the first week, and the year's total at that rate.

```
2 divided by 7 is about 0.2857
0.2857 times 365 = 104.3
```

About 0.29 kg a day, and about 104 kg a year at that rate. The report's arithmetic is right.

What it does not establish. The 2 kg is an average over one week, and it says nothing about the rate now. It is also the wrong week to project from. Hall (2008) notes that glycogen's "effects on weight loss typically occur within the first week". Glycogen is held in the body with water. So part of the first week's loss is glycogen and its water, which will not keep coming off week after week.

Hall and colleagues (2012) add that "weight change will slow over time". So the first-week rate overstates every later week. The year's total cannot be found by multiplying one week's average. It needs the rate at each point along the way.

12. What to compute: the average rate over the one day between the two weighings.

```
76.2 minus 76.2 = 0
0 divided by 1 = 0
```

Zero kg a day, over that one day. The arithmetic is right.

What it does not establish. First, it is not dW/dt. It is an average over one day, and one day sits inside the normal swings of body water. Two equal readings can come from a falling trend with a small swing up on the second day.

Second, even a true zero for weight would not mean zero for stores. Hall and colleagues (2012) point out that weight change "may not directly represent energy imbalances, particularly over the short term". A zero dW/dt and a zero dES/dt are different claims.

Third, a zero rate at one instant is not a steady weight. The rate may be passing through zero on its way from falling to rising.

What to write instead. Record both weights. Ask for weighings over several weeks, and take the rate from averages over that time.

3 · The few rules that do the differentiating

EXERCISE 1. Three minutes, no jargon, one quotable sentence.

Start with what slowing means. Weight is still going down. It is going down more slowly than before.

Then say why that is expected. A smaller body burns less energy a day. Appetite also rises as weight falls. So the same drug that caused a loss at the start causes a smaller loss later. Hall and colleagues (2012) put it this way: weight change "will slow over time" because expenditure falls as weight falls. That happens even if the drug keeps working exactly as it did on day one.

Then say what would tell you the effect wears off. You would need a comparison: people on the drug against people on a dummy pill, over the same months.

The sentence to quote: "Weight loss slows on any treatment that keeps working, because a smaller body burns less and appetite rises as weight falls. A slowing curve does not show the drug has stopped working."

Calibrate it. The slowing is well established. Whether this drug's effect fades is a separate question the curve cannot answer.

1. The constant rule for the first. The power rule for the other two.

```
d/dx (9) = 0
d/dx (x^6) = 6 x^5
d/dx (x) = 1 times x^0 = 1
```

2. Keep the number in front, and take the derivative of the power.

```
d/dx (4 x^3) = 4 times 3 x^2 = 12 x^2
d/dt (-2 t^5) = -2 times 5 t^4 = -10 t^4
```

3. Take each term on its own, then combine.

```
d/dx (3 x^4) = 3 times 4 x^3 = 12 x^3
d/dx (2x) = 2 times 1 = 2
d/dx (6) = 0
f'(x) = 12 x^3 minus 2 + 0 = 12 x^3 minus 2
```

4. Differentiate term by term.

```
f'(t) = 2 times 3 t^2 minus 1 = 6 t^2 minus 1
```

Put in t = 2.

```
6 times 2 times 2 minus 1 = 23
```

f′(2) = 23.

5. Differentiate twice.

```
f'(t) = 3 t^2 minus 12t
f''(t) = 6t minus 12
```

At t = 1.

```
3 times 1 minus 12 times 1 = -9
6 times 1 minus 12 = -6
```

At t = 3.

```
3 times 9 minus 12 times 3 = -9
6 times 3 minus 12 = 6
```

The rate is minus 9 at both times, so f is falling at the same speed at both. At t = 1 the second derivative is negative: the rate is getting more negative, so the fall is speeding up. At t = 3 it is positive: the rate is rising towards zero, so the fall is slowing.

Same rate, opposite futures. The first derivative alone could not tell them apart.

6. The constant goes to zero. The term in x keeps its number.

```
dy/dx = 0 minus 0.4 times 1 = -0.4
d^2y/dx^2 = 0
```

The first derivative is the slope, minus 0.4, at every x. The second derivative is zero, because a straight line's rate never changes.

7. The derivative of a straight line is its slope, the same at every weight. So the change in expenditure is the slope times the change in weight. The weight change is minus 8 kg.

```
20 times -8 = -160
30 times -8 = -240
```

Expenditure falls by 160 to 240 kcal a day along the line.

For a straight line this is exact. For a curve it would hold only for a small change.

8. Write the rule, then differentiate term by term.

```
BMR(W) = 15.057W + 692.2
dBMR/dW = 15.057 times 1 + 0 = 15.057
```

15.057 kcal a day per kg.

Now the two weights.

```
15.057 times 60 = 903.42
903.42 plus 692.2 = 1,595.62
15.057 times 70 = 1,053.99
1,053.99 plus 692.2 = 1,746.19
1,746.19 minus 1,595.62 = 150.57
15.057 times 10 = 150.57
```

The two agree exactly, because the rule is a straight line. The 692.2 cancels in the subtraction, as the constant rule says it must.

9. Write TEE as a rule in W.

```
TEE(W) = 1.75 times (15.057W + 692.2)
```

The constant multiple rule keeps the 1.75 outside. The derivative of the bracket is 15.057, from the last problem.

```
dTEE/dW = 1.75 times 15.057 = 26.35
```

About 26.35 kcal a day per kg. That sits inside Hall and Guo's range of 20 to 30.

What it shows: the two figures are the same kind of quantity, in the same unit, and of the same size.

What it does not show. It does not confirm either figure. The 1.75 is a PAL chosen for an example, and here it is held fixed as weight changes. The Table 5.2 line is for one age band of men. The Indian Council of Medical Research's National Institute of Nutrition (ICMR-NIN) says in 2020 that these equations can overestimate BMR by 10 to 12% for Indians. Scaling BMR by 0.9 scales the slope by 0.9, to about 23.7, by the constant multiple rule. Hall and Guo's slope comes from other people and other methods. Agreement in size is a reason to take both seriously, not proof that either is right for a given patient.

10. Differentiate twice.

```
W'(t) = -0.15 + 0.0005 times 2t = -0.15 + 0.001t
W''(t) = 0.001
```

At day 60.

```
-0.15 + 0.001 times 60 = -0.09
```

W′(60) = minus 0.09 kg a day. W″ = 0.001 kg a day per day. At day 60 weight is falling by 0.09 kg a day, and the fall is slowing.

W′ reaches zero where minus 0.15 plus 0.001t is zero. Add 0.15 to both sides, then divide by 0.001 (Book 0, C2).

```
0.001t = 0.15
0.15 divided by 0.001 = 150
```

Day 150. That is where this made-up curve stops falling. It says nothing about when a real person's weight would level off.

11. The third step broke. The derivative of a constant is zero, not the constant.

```
d/dx (5) = 0
f'(x) = 6x + 0 = 6x
f'(0) = 6 times 0 = 0
```

f′(0) is 0, not 5.

Check at one point with the shrinking-interval method. f(0) = 5 and f(0.01) = 5.0003. So the average over that step is 0.0003 divided by 0.01, which is 0.03. That is near 0, not near 5.

Why the wrong answer looks reasonable. The power rule was used correctly on the first term. The constant was then treated as if it too were being differentiated into itself. A 5 that sits still has no rate.

12. The second step broke. The power rule is for the input raised to a fixed number, like t^2. In 2^t the fixed number is at the bottom and the input is up in the power. The rule does not apply.

Check at t = 1 by shrinking the interval, as in the last section. Take h = 0.001.

```
g(1) = 2
g(1.001) = 2.0013868
2.0013868 minus 2 = 0.0013868
0.0013868 divided by 0.001 = 1.3868
```

The average rate near t = 1 is about 1.39, not 1. The rule gave a wrong answer.

Why the wrong answer looks reasonable. It has the exact form of the power rule, with the power brought down and one taken off. The two symbols swapped roles.

13. The third step broke. The derivative of a product is not the product of the derivatives.

Multiply the product out first. By Book 0, A6, multiplying powers of the same base adds the exponents.

```
h(t) = t^(2 + 3) = t^5
h'(t) = 5 t^4
h'(1) = 5 times 1 = 5
```

h′(1) is 5, not 6.

The rule that handles products without multiplying out is the product rule. It comes in the rung-2 book of this subject.

Why the wrong answer looks reasonable. The sum rule works term by term, so it feels natural that a product would work factor by factor. Both factor derivatives are right. Only the way they were combined is wrong.

14. The third step broke. It reads the sign of the second derivative as the direction of the weight. The direction is the sign of the first derivative.

```
W’(t) = -0.3 + 0.004t
W’(1) = -0.3 + 0.004 = -0.296
W’(60) = -0.3 + 0.004 times 60 = -0.06
```

The first derivative is negative from day 1 to day 60. Weight falls all the way. The positive second derivative says the fall is slowing: minus 0.296 kg a day at day 1, minus 0.06 at day 60.

Why the wrong answer looks reasonable. “Positive derivative means rising” is true. It is true of W″ too: what is rising is the rate, not the weight.

15. What to compute: how far expenditure has fallen along a straight line with that slope. Then what that means for the gap between intake and expenditure.

```
20 times -10 = -200
30 times -10 = -300
```

Expenditure has fallen by 200 to 300 kcal a day along the line.

Now the reasoning. To keep losing at the same speed she needs the same gap between intake and expenditure as at the start. Expenditure has fallen by 200 to 300. So intake must fall by a further 200 to 300 kcal a day beyond her first cut. The colleague’s sentence measures from the wrong starting point: it forgets the first cut.

What this does not establish. The slope of 20 to 30 is Hall and Guo’s approximate figure. One person’s slope may differ. A straight line also leaves out the extra fall in expenditure after weight loss that Book 1 described (Book 1, section 9).

And it leaves out appetite. Polidori and colleagues (2016) found eating rose by about 100 kcal a day per kg of weight lost. That pushes intake up by far more than the expenditure line moves. Holding intake down is the harder part, and this sum says nothing about it.

16. What to compute: the change each month, which is the rate over each month. Then the change in that change, which is the discrete version of the second derivative, as in the plain-terms text.

```
97 minus 100 = -3
94.5 minus 97 = -2.5
92.5 minus 94.5 = -2
91 minus 92.5 = -1.5
90 minus 91 = -1
89.3 minus 90 = -0.7
88.8 minus 89.3 = -0.5
88.5 minus 88.8 = -0.3
```

Weight fell every month. Now the change in the monthly change.

```
-2.5 minus -3 = 0.5
-2 minus -2.5 = 0.5
-1.5 minus -2 = 0.5
-1 minus -1.5 = 0.5
-0.7 minus -1 = 0.3
-0.5 minus -0.7 = 0.2
-0.3 minus -0.5 = 0.2
```

Every one is positive. The loss was slowing from month 1, not from month 6. Nothing changes at month 6.

What this does not establish. A slowing loss is what happens anyway. Hall and colleagues (2012) write that weight change "will slow over time due to passive compensatory changes in energy expenditure". So the curve is what a medicine that keeps working would also produce.

It does not show the medicine stopped working, and it does not show it kept working. That needs a comparison group on a dummy pill over the same months. It also needs to know whether people kept taking the medicine.

4 · The number e, natural logarithms and change proportional to amount

EXERCISE 1. Set the curve equal to one twentieth of its start. That is 5%, since 5 divided by 100 is 1/20.

```
A e^(-kt) = A divided by 20    (the condition)
e^(-kt) = 1/20    (divide both sides by A)
-kt = ln(1/20)    (take ln of both sides)
-kt = -ln 20    (ln of one over a number is minus its ln)
t = ln 20 divided by k    (divide both sides by -k)
```

Then put the number in.

```
ln 20 = 2.996
```

So t is about 3.00 divided by k.

A cancels in the second line. The curve takes the same time to fall to a twentieth of any starting value, large or small. That is the property that makes one number, a half-time or a 95% time, enough to describe the whole curve. It is also why the number stops meaning anything when the curve is not one exponential.

1. Key each one in.

```
e^1 = 2.718
e^2 = 7.389
e^(-1) = 0.368
ln 1 = 0
ln 10 = 2.303
ln(e^3) = 3
```

Two of these need no calculator. The ln of 1 is 0, because e^0 is 1. The ln of e^3 is 3, because ln undoes a power of e. Notice that $e^{(-1)}$ is a positive number below one. A minus sign in an exponent makes a number small, not negative.

2. First the two parts.

```
ln 4 = 1.386
ln 5 = 1.609
1.386 plus 1.609 = 2.995
ln 20 = 2.996
```

The two agree to the last place, where rounding each part moved the sum by 0.001.

Now the half.

```
ln 0.5 = -0.693
ln 2 = 0.693
```

Equal size, opposite sign. The reason is that 2 times 1/2 is 1, and ln 1 is 0, so the two logarithms must add to zero.

3. The rate of A $e^{(kt)}$ is k times A $e^{(kt)}$. Here A is 40 and k is 0.05.

```
0.05 times 40 = 2
```

So dy/dt = 2 $e^{(0.05t)}$.

At t = 0, e^0 is 1.

```
y = 40
dy/dt = 2
2 divided by 40 = 0.05
```

At t = 10, the exponent is 0.05 times 10, which is 0.5.

```
e^0.5 = 1.6487
y = 40 times 1.6487 = 65.95
dy/dt = 2 times 1.6487 = 3.297
3.297 divided by 65.95 = 0.05
```

The rate grew from 2 to about 3.3, but the rate divided by the amount stayed at 0.05. That steady ratio is k, and it is what makes the curve exponential.

4. The half-time is ln 2 divided by k, with k = 0.1.

```
0.693 divided by 0.1 = 6.93
```

The time to 5% is ln 20 divided by k.

```
2.996 divided by 0.1 = 29.96
```

Now y at each time. At the half-time y is half of 200, which is 100. At the 5% time it is one twentieth of 200.

```
200 divided by 2 = 100
200 divided by 20 = 10
```

Check the second on the calculator: 0.1 times 29.96 is 2.996, and $e^{(-2.996)}$ is 0.0500. So y is 200 times 0.0500, which is 10. The half-time is 6.93 and the 5% time is 29.96, in whatever unit t was measured in.

5. Turn the half-time formula round: k is ln 2 divided by the half-time.

```
0.693 divided by 14 = 0.0495
```

So k is 0.0495 per day.

After 28 days, two half-times have passed, so a half of a half is left, one quarter. Check it with the formula.

```
0.0495 times 28 = 1.386
e^(-1.386) = 0.250
```

A quarter, 0.250.

After 10 days, less than one half-time has passed, so more than half is left.

```
0.0495 times 10 = 0.495
e^(-0.495) = 0.610
```

About 0.610, or 61% of the start.

6. Divide both sides by 100.

```
e^(-0.2t) = 0.3
```

Take ln of both sides.

```
-0.2t = ln 0.3 = -1.204
```

Divide both sides by -0.2.

```
1.204 divided by 0.2 = 6.02
```

So t = 6.02. Check: $e^{(-0.2 \text{ times } 6.02)}$ is $e^{(-1.204)}$, which is 0.300, and 100 times 0.300 is 30.

7. Subtract each reading from the next.

```
70 minus 60 = 10
75 minus 70 = 5
77.5 minus 75 = 2.5
```

Divide each difference by the one before.

```
5 divided by 10 = 0.5
2.5 divided by 5 = 0.5
```

A steady ratio of 0.5, so a ceiling can be worked out (Book 0, C6). The next difference is 2.5 times 0.5, and the gap left is that divided by one minus the ratio.

```
2.5 times 0.5 = 1.25
1.25 divided by (1 minus 0.5) = 2.5
77.5 plus 2.5 = 80
```

The level is 80. The gap to 80 runs 20, 10, 5, 2.5: it halves every step. So the half-time is one step, 2 units of time. The rate constant follows.

```
0.693 divided by 2 = 0.3465
```

k is about 0.347 per unit of time. The same comes from the ratio: minus ln 0.5 is 0.693, and 0.693 divided by a step of 2 is 0.3465.

8. Eventual change: 1 kilogram for every 100 kilojoules a day.

```
500 divided by 100 = 5
```

About 5 kilograms, eventually.

Now the supposed exponential. A half-time of 1 year gives k = 0.693 per year. Six months is 0.5 years.

```
0.693 times 0.5 = 0.3465
e^(-0.3465) = 0.7072
1 minus 0.7072 = 0.2928
5 times 0.2928 = 1.46
```

About 1.46 kilograms by 6 months. Two years is two half-times, so three quarters of the way.

```
5 times 0.75 = 3.75
```

About 3.75 kilograms by 2 years.

Whose figures are these? The 100 kJ a day per kilogram and the one-year half are the paper's, both rounded. The 500 kJ is yours. The exponential is your assumption, so 1.46 and 3.75 are yours too. The paper does not say its curve is one exponential.

9. The gap at the start is 100 minus 75, which is 25 kilograms.

First exponential: 95% at 3 years, so k is ln 20 divided by 3.

```
2.996 divided by 3 = 0.999
e^(-0.999) = 0.368
25 times 0.368 = 9.2
75 plus 9.2 = 84.2
```

About 84.2 kilograms at one year.

Second exponential: a half-time of 1 year, so after 1 year half the gap is left.

```
25 divided by 2 = 12.5
75 plus 12.5 = 87.5
87.5 minus 84.2 = 3.3
```

About 87.5 kilograms, 3.3 kilograms higher. Both curves match one of the paper's rounded figures. They disagree at one year by more than 3 kilograms, which is why neither should be quoted as the model's prediction.

10. The rate of a closing gap is minus k times the gap. Here k is 0.693 per year and the gap is 25 kilograms.

```
0.693 times 25 = 17.325
```

At the start the weight falls at about 17.3 kilograms a year. Turn that into a daily rate.

```
17.325 divided by 365 = 0.0475
```

About 0.0475 kilograms a day, or 47.5 grams a day.

Over the first year, half the gap closes, which is 12.5 kilograms. So the average rate over that year is 12.5 kilograms a year.

The rate at the start, 17.3, is well above the first-year average, 12.5. An exponential closes fastest at the start. So a first-year average is not the rate in the first weeks. And a rate from the first weeks is not the rate for the year.

11. The second line broke. The half-time is ln 2 divided by k, with the natural logarithm, not $\log_{10} 2$.

```
0.693 divided by 0.1 = 6.93
```

It halves every 6.93 days, more than twice as long as the worked answer says.

The half-time comes from solving $e^{(-kt)} = 1/2$. Only ln undoes a power of e, because k was defined with e as the base. Using $\log_{10}$ answers a different question: when does $10^{(-kt)}$ reach one half?

Why the wrong answer looks reasonable: the calculator key marked log is $\log_{10}$, and it is the logarithm Book 0 taught. The arithmetic after the second line is right. Check any half-time by putting it back: $e^{(-0.1 \text{ times } 3.01)}$ is 0.74, not 0.5.

12. The second line broke. The share still to come is $e^{(-kt)}$, not the share already reached. At 95% of the way, 5% is still to come.

```
e^(-kt) = 0.05
-kt = ln 0.05 = -2.996
t = 2.996 divided by 0.693 = 4.32
```

About 4.3 years, not 27 days.

The worked answer found the time for the first 5% of the change. That is a real number, but not the one asked for.

Why it looks reasonable: 0.95 is the number in the question, so it goes straight into the equation. The check that catches it: 27 days is far less than one half-time, which here is one year, and 95% cannot come before 50% does.

13. The second line broke. The power rule (section 3) is for t raised to a fixed power, where the variable is in the base. Here the variable is in the exponent, and the base is the fixed number e. That is a different kind of function with a different rule.

The rate of A $e^{(kt)}$ is k times A $e^{(kt)}$.

```
0.05 times 40 = 2
dy/dt = 2 e^(0.05t)
at t = 0, dy/dt = 2
```

The curve starts rising at 2 per unit of time, not flat.

A check that catches it without any rule: step forward a little and divide. At t = 0.001, y is 40 times $e^{(0.00005)}$, which is about 40.002. The change is 0.002 over 0.001, a rate of 2.

Why it looks reasonable: “bring the power down and take one off” is the first rule most people learn. And 0.05t sits where a power usually sits. The answer even looks like calculus.

14. The second line broke. The rate constant is 0.693 per year, and 30 is in days. The units have to match before they are multiplied.

Turn 30 days into years.

```
30 days = 0.0822 years
0.693 times 0.0822 = 0.057
e^(-0.057) = 0.945
```

After a month, 94.5% of the gap is still there. About 5.5% of the change has happened, not all of it.

Why it looks reasonable: k is often written as a bare number, 0.693, with its unit dropped. The calculation after that is correct arithmetic on the wrong product. The check that catches it: a one-year half-time cannot finish in a month.

15. What to compute: what a one-year half-time gives at three years, and whether any one exponential could give both of the paper's figures.

Three years is three half-times.

```
0.5 times 0.5 times 0.5 = 0.125
1 minus 0.125 = 0.875
```

So 87.5%, as the colleague says. Now the ratio test. For one exponential, the time to 95% is 4.32 half-times. The paper's figures give 3 years over 1 year, which is 3. No single exponential gives both.

What this does not establish. It does not show the paper contradicts itself. The paper never claims its curve is one exponential. It reports two rounded outputs of its model, and a model's curve can slow in other ways. It is also possible that "about 1 year" rounds something nearer 8 months.

What it does establish is narrower and useful. You cannot rebuild the paper's curve from its two figures by assuming one exponential. If you need a figure between them, you need the model's own output.

16. What to compute: first, how the 16 was reached. Then what an exponential slowing would give instead.

The 16 is 4 kilograms times four quarters of a year.

```
4 times 4 = 16
```

That is a straight line, the same loss every quarter. It contradicts the first half of the sentence, which says the loss slows.

Now suppose one exponential with a half-time of 1 year. By three months, a quarter of a year, this share of the eventual change has come.

```
0.693 times 0.25 = 0.1733
e^(-0.1733) = 0.841
1 minus 0.841 = 0.159
4 divided by 0.159 = 25.2
25.2 times 0.5 = 12.6
```

An eventual loss near 25 kilograms, and about 12.6 by the end of the year. Take instead the exponential that is 95% of the way at 3 years, with k = 0.999 per year. At one year this curve is 63.2% of the way (the table above).

```
0.999 times 0.25 = 0.2498
e^(-0.2498) = 0.779
1 minus 0.779 = 0.221
4 divided by 0.221 = 18.1
18.1 times 0.632 = 11.4
```

About 11.4 kilograms by the end of the year. Either way, the year's figure is well under 16.

What this does not establish. It does not establish that this patient follows either curve. Both assume intake stays cut by the same amount, and the curve's timescale from Hall's model is for an average adult, not for this person. It does not establish that the first three months were typical: early loss includes water and glycogen, not only tissue. And it says nothing about what the patient should do. The only firm conclusion is the arithmetic one: a loss that slows cannot be extended as a straight line.

5 · The integral of a rate is the change in the stock

EXERCISE 1. As rates, with EE for what the paper calls EO:

```
dES/dt = EI minus EE
```

In words: the rate of change of the stored energy ES with respect to time t equals intake minus expenditure.

Each symbol and its unit:

symbol	what it stands for	unit
ES	the energy stored in the body	kcal
t	time	days
dES/dt	the rate of change of ES	kcal a day
EI	energy intake, a rate	kcal a day
EE	energy expenditure, a rate	kcal a day

As an integral, by the net change theorem:

```
ES(b) minus ES(a) = ∫ from a to b of (EI minus EE) dt
```

The left side is a change in the stores, in kilocalories. The right side is kilocalories a day times days, also kilocalories. The two sides carry the same unit, which is the check that the equation is written right.

As section 1 said, the paper's own shorthand is ES = EI minus EO, and there ES is the rate. In this book ES is the store and dES/dt its rate.

1. The lower limit is 2 and the upper limit is 7. The integrand is 3t plus 1. The variable is t.

The integrand is a straight line, so use the trapezium. Work out the two end rates.

```
3 times 2 plus 1 = 7
3 times 7 plus 1 = 22
```

Average them and multiply by the width, 7 minus 2.

```
7 plus 22 = 29
29 divided by 2 = 14.5
7 minus 2 = 5
14.5 times 5 = 72.5
```

The integral is 72.5.

2. Each is a steady rate, so each is a rectangle: the rate times the width.

```
4 times 6 = 24
-4 times 6 = -24
```

The first is 24. The second is minus 24, because its rectangle sits below the axis.

3. At t = 0 the rate is 0, and at t = 8 it is 5 times 8, which is 40. The shape is a triangle with base 8 and height 40.

```
5 times 8 = 40
0.5 times 8 times 40 = 160
```

The integral is 160. The trapezium gives the same: the average of 0 and 40 is 20, and 20 times 8 is 160.

4. The rate is 6 at t = 0 and minus 4 at t = 10. It crosses zero at t = 6.

Above the axis is a triangle from 0 to 6, with height 6. Below is a triangle from 6 to 10, with depth 4.

```
0.5 times 6 times 6 = 18
0.5 times 4 times 4 = 8
18 minus 8 = 10
```

The integral is 10. Check with the trapezium, keeping the sign.

```
6 plus -4 = 2
2 divided by 2 = 1
1 times 10 = 10
```

The total area, all counted positive, is 18 plus 8, which is 26. The two differ because the integral subtracts the part below the axis, and the total area adds it.

5. First the integral. The end rates are 6 and 14.

```
2 times 1 plus 4 = 6
2 times 5 plus 4 = 14
6 plus 14 = 20
20 divided by 2 = 10
10 times 4 = 40
```

The integral is 40, over a width of 5 minus 1, which is 4. The average value is the integral divided by the width.

```
40 divided by 4 = 10
```

The average is 10. For a straight line it is halfway between the ends, and halfway between 6 and 14 is 10.

6. The change from 0 to 10 is the change from 0 to 4 plus the change from 4 to 10. So subtract.

```
50 minus 18 = 32
```

The integral from 4 to 10 is 32.

Running from 10 back to 4 undoes the same change, so it has the opposite sign. It is minus 32. Swapping the limits always flips the sign.

7. A steady uncertainty in a rate, held over a time, is a rectangle.

```
100 times 14 = 1400
```

More than 1,400 kilocalories over the 14 days. The unit is kilocalories a day times days, which is kilocalories.

The method gives one average rate for the whole period (Book 1, section 3). So the uncertainty in that average is multiplied by the 14 days. It is “more than” because the paper says more than 100 a day.

8. A steady deficit is a rectangle.

```
480 times 365 = 175200
```

175,200 kilocalories.

A straight-line deficit is a trapezium.

```
480 plus 240 = 720
720 divided by 2 = 360
360 times 365 = 131400
```

131,400 kilocalories. The average daily deficit is the middle number, 360 kilocalories a day.

The shrinking deficit gives about a quarter less over the year. Which shape is right is not something this arithmetic can tell you. Only the 480 is from the paper.

9. The average rate is the change divided by the time.

```
-9000 divided by 30 = -300
```

Minus 300 kilocalories a day on average.

For a straight line, the average is halfway between the ends. So the two ends add to twice the average.

```
-300 times 2 = -600
-600 minus -500 = -100
```

The rate ended at minus 100 kilocalories a day. Check: the average of minus 500 and minus 100 is minus 300, and minus 300 times 30 is minus 9,000.

10. The fifth line broke. The two areas have opposite signs. The part below the axis is a loss and the part above is a gain, so they are combined by subtraction.

```
-4000 plus 1000 = -3000
```

The stores fell by 3,000 kilocalories. Check with the trapezium, keeping the signs.

```
-400 plus 200 = -200
-200 divided by 2 = -100
-100 times 30 = -3000
```

Why it looks reasonable: both areas are right, and the answer is called a fall, which feels as if the sign has been handled. But the last 10 days put 1,000 kilocalories back. Adding them to the loss counted them the wrong way.

11. The second line broke. The rate is per day, but the limits run from 0 to 4 in weeks. The time in the limits has to be in the same unit as the time in the rate.

```
4 times 7 = 28
300 times 28 = 8400
```

The stores rose by 8,400 kilocalories.

Why it looks reasonable: “4 weeks” became “4” and the unit was dropped. Kilocalories a day times weeks is not kilocalories, and the unit check would have caught it.

12. The third line broke. It averages the three readings as if each stood for the same length of time. The first gap is 10 days and the second is 30 days.

Work each stretch as its own trapezium, then add.

```
400 plus 300 = 700
700 divided by 2 = 350
350 times 10 = 3500
300 plus 0 = 300
300 divided by 2 = 150
150 times 30 = 4500
3500 plus 4500 = 8000
```

The total deficit is 8,000 kilocalories. The average rate is that divided by 40 days.

```
8000 divided by 40 = 200
```

200 kilocalories a day, not 233.3.

Why it looks reasonable: averaging readings is what an average usually means. That works only when the readings are equally spaced. Here the lowest reading covers the longest stretch, so the plain average overstates the deficit.

13. What to compute: whether the 12,600 follows from the rate, and how large the error could be beside it.

Twelve weeks is 84 days.

```
12 times 7 = 84
-150 times 84 = -12600
```

The arithmetic holds. An average rate times the time gives the change, and the unit is kilocalories.

Now the error. The paper says the error in the rate can reach 1,000 kilocalories a day. That is more than six times the imbalance being reported.

```
1000 divided by 150 is about 6.7
```

What this does not establish. It does not establish that each participant lost 12,600 kilocalories. The figure is an average rate times a time, so it is an average change, not each person’s. It does not establish that the imbalance was measured well enough to mean anything, since the possible error is several times the signal. And it says nothing about body weight. That needs to know what the lost stores were made of.

What to ask for: how the imbalance was measured. And ask whether the change in stores was measured directly, for example from body composition, rather than worked out from rates.

14. What to compute: the change in stores over the two months, from the rate. Take two months as 60 days, and take the fall as a straight line, since only the two ends are given.

The rate is above zero the whole time, falling from 400 to 0. The shape is a triangle.

```
0.5 times 60 times 400 = 12000
```

Her stores rose by about 12,000 kilocalories. A falling surplus is still a surplus. The stores rose more slowly each day, but they rose every day until the rate reached zero.

What this does not establish. It does not establish the exact figure. The straight line is an assumption, and a surplus that fell fast at first would give less. It does not tell you her weight change. That needs what the stored energy was made of, and a scale also sees water. And it says nothing about why the surplus fell, or what it will do next.

What to say instead: “Her surplus has been shrinking, so her stores have been rising more slowly. They have not been falling.”

6 · Undoing a derivative: antiderivatives and the fundamental theorem

EXERCISE 1. Find an antiderivative, check it, then take the end minus the start.

```
F(t) = D0 e^(-kt) divided by -k    (the rule for e^(kt), with -k in place of k)
the rate of F = -k times D0 e^(-kt) divided by -k = D0 e^(-kt)    (check by differenti-
ating back)
total = F(T) minus F(0)    (the fundamental theorem)
F(0) = D0 divided by -k    (e^0 is 1)
total = D0 e^(-kT) divided by -k minus D0 divided by -k    (substitute)
total = D0 times (1 minus e^(-kT)) divided by k    (multiply top and bottom by -1 and
collect)
```

As T grows, $e^{(-kT)}$ shrinks towards zero. So 1 minus $e^{(-kT)}$ climbs towards 1, and the total climbs towards D0 divided by k. It never gets there and never passes it.

For example, take D0 = 730 and k = 0.693, as in the illustration.

```
730 divided by 0.693 = 1053.4
```

The total over all time is at most 1053.4, however long the deficit runs.

1. Use the table’s rules term by term.

```
6 gives 6t
4t gives 4 times t^2 divided by 2, which is 2 t^2
3 t^2 gives 3 times t^3 divided by 3, which is t^3
t^3 plus 2t gives t^4 divided by 4 plus t^2
```

Now differentiate each back with the power rule.

```
the rate of 6t is 6
the rate of 2 t^2 is 2 times 2t, which is 4t
the rate of t^3 is 3 t^2
the rate of t^4 divided by 4 plus t^2 is t^3 plus 2t
```

Each rate comes back. Any of these plus a constant would do as well.

2. An antiderivative of $3\,t^2$ is t^3. Take it at the upper limit minus the lower limit.

```
3^3 = 27
1^3 = 1
27 minus 1 = 26
```

The integral is 26.

3. An antiderivative of $e^{(3t)}$ is $e^{(3t)}$ divided by 3. At t = 2 the exponent is 6. At t = 0 it is 0, and e^0 is 1.

On the calculator, e^6 is 403.43. Take the antiderivative at 2 minus the antiderivative at 0.

```
403.43 divided by 3 minus 1 divided by 3 = (403.43 minus 1) divided by 3
403.43 minus 1 = 402.43
402.43 divided by 3 = 134.14
```

The integral is 134.14.

4. An antiderivative of 10 $e^{(-0.5t)}$ is 10 $e^{(-0.5t)}$ divided by -0.5, which is -20 $e^{(-0.5t)}$. Check: its rate is -0.5 times -20 $e^{(-0.5t)}$, which is 10 $e^{(-0.5t)}$.

At t = 4 the exponent is -2. At t = 0, e^0 is 1.

```
e^(-2) = 0.1353
-20 times 0.1353 = -2.706
-20 times 1 = -20
-2.706 minus -20 = 17.294
```

The integral is about 17.29.

5. With t^2:

```
5^2 = 25
2^2 = 4
25 minus 4 = 21
```

With t^2 plus 5:

```
25 plus 5 = 30
4 plus 5 = 9
30 minus 9 = 21
```

Both give 21. The 5 is added at the upper limit and again at the lower limit, and the subtraction takes it away. That is why the plus C is left out when you work out an integral between two limits.

6. The total from 0 to T is 10 times (1 minus $e^{(-0.5T)}$) divided by 0.5. As T grows, $e^{(-0.5T)}$ shrinks towards zero.

```
10 divided by 0.5 = 20
```

Over all time the total approaches 20.

It reaches 95% of 20 when 1 minus $e^{(-0.5T)}$ is 0.95. That is when $e^{(-0.5T)}$ is 0.05, which is one twentieth. From section 4, that happens at ln 20 divided by k.

```
2.996 divided by 0.5 = 5.992
```

At about T = 5.99.

7. The deficit is 480 minus 0.5t, with t in days. Take it term by term.

```
0.5 divided by 2 = 0.25
F(t) = 480t minus 0.25 t^2
```

Check: the rate of F is 480 minus 2 times 0.25t, which is 480 minus 0.5t. Now F at 200 minus F at 0. F at 0 is 0.

```
480 times 200 = 96000
200 times 200 = 40000
0.25 times 40000 = 10000
96000 minus 10000 = 86000
```

The total deficit is 86,000 kilocalories.

Check with the trapezium. On day 200 the deficit is 480 minus 100, which is 380.

```
480 plus 380 = 860
860 divided by 2 = 430
430 times 200 = 86000
```

The same 86,000 kilocalories.

8. Steady: a rectangle.

```
480 times 90 = 43200
```

Halving every 365 days: the rate constant is ln 2 divided by 365, per day.

```
0.693 divided by 365 = 0.001899
0.001899 times 90 = 0.1709
```

On the calculator, $e^{(-0.1709)}$ is 0.8429. The total is 480 times (1 minus 0.8429) divided by 0.001899.

```
1 minus 0.8429 = 0.1571
480 times 0.1571 = 75.408
75.408 divided by 0.001899 = 39709.3
```

About 39,700 kilocalories.

```
39709.3 divided by 43200 = 0.919
```

The halving deficit gives about 92% of the steady one over 90 days. Over three months the shrinking hardly shows. It grows with time, which is why a rule tested only in the first weeks can look better than it is.

9. Run the total backwards. Over the first year the total is D0 times (1 minus $e^{(-0.693)}$) divided by 0.693. On the calculator, $e^{(-0.693)}$ is 0.5001, so the bracket is 0.4999.

```
D0 times 0.4999 divided by 0.693 = 400
```

Multiply both sides by 0.693, then divide both sides by 0.4999.

```
400 times 0.693 = 277.2
277.2 divided by 0.4999 = 554.5
```

D0 is about 554.5 megajoules a year. Divide by 365 for a daily figure.

```
554.5 divided by 365 = 1.519
```

About 1.52 megajoules a day at the start.

10. The third line broke. The theorem takes F at the upper limit minus F at the lower limit, so it is F(3) minus F(0).

```
27 minus 0 = 27
```

The integral is 27.

Why it looks reasonable: the antiderivative is right, and so are both values of F. Only the order is wrong. The check that catches it: the rate 12 minus 2t is above zero all the way from 0 to 3. So its total must be above zero.

11. The second line broke. Raising the power by one is half the rule. The other half divides by the new power. An antiderivative of t^2 is t^3 divided by 3.

```
27 divided by 3 = 9
```

The integral is 9.

Differentiating back catches it at once. The rate of t^3 is 3 t^2, not t^2. The broken answer is three times too big.

Why it looks reasonable: the power went up by one, which is the part of the rule people remember. Nothing on the page is checked.

12. The second line broke. The rate constant is 0.693 per year, and 30 is in days. The deficit is also per day. Put everything in days first.

```
0.693 divided by 365 = 0.001899
0.001899 times 30 = 0.05697
```

On the calculator, $e^{(-0.05697)}$ is 0.9446.

```
1 minus 0.9446 = 0.0554
480 times 0.0554 = 26.592
26.592 divided by 0.001899 = 14003.2
```

About 14,000 kilocalories over the month. A steady 480 a day would give 14,400, so the shrinking barely shows yet.

Why it looks reasonable: the formula is the right one, and each line after the second is correct arithmetic. But a total of 693 kilocalories over a month is less than two days of the starting deficit. A deficit that halves over a year cannot have shrunk that fast.

13. What to compute: the total a shrinking deficit adds up to over all time, against the total of a steady one.

Take a deficit D0 $e^{(-kt)}$ that never reaches zero. Its total from 0 to T is D0 times (1 minus $e^{(-kT)}$) divided by k, and that approaches D0 divided by k. Put in the numbers from this section's illustration: 730 megajoules a year at the start, halving each year.

```
730 divided by 0.693 = 1053.4
```

However long it runs, the total stays below 1,053.4 megajoules. The steady deficit, the old rule's picture, passes that within a year and a half.

```
730 times 1.5 = 1095
```

So "never reaches zero" does not mean "adds up to anything". A rate can stay above zero for ever and still have a total that stops growing.

What this does not establish. It does not establish that a real deficit shrinks as an exponential. That was an assumption, chosen to show the arithmetic. It says nothing about kilograms, which need what the lost tissue was made of. And it does not say the loss stops at a fixed date. It says the total levels off, which is a different claim.

14. What to compute: whether the gap between a steady deficit and a shrinking one is the same at every time. Use the made-up halving deficit from this section, 730 megajoules a year at the start, halving each year. It is not the paper's model, but it shows how such a gap behaves.

Take the shrinking total as a share of the steady total, at 1, 3 and 10 years. At 3 years, $e^{(-2.079)}$ is 0.125, so the shrinking total is 1053.4 times 0.875, which is 921.7. At 10 years, $e^{(-6.93)}$ is 0.001, so the shrinking total is 1053.4 times 0.999.

```
526.6 divided by 730 = 0.721
921.7 divided by 2190 = 0.421
1053.4 times 0.999 = 1052.3
730 times 10 = 7300
1052.3 divided by 7300 = 0.144
```

For this made-up deficit, the share falls from about 72% at one year to 42% at three years and 14% at ten. The gap grows with time. So no one factor fits every year.

What this does not establish. It does not give the paper's own correction for any year but the first, for the one modelled man. The "about 100%" is a first-year comparison. It does not establish that halving is right even at one year for a different person or deficit. And the made-up halving deficit is not the paper's model, so its percentages are not the paper's.

What to say instead: the old rule's error is not a fixed factor. It grows with time, because a shrinking deficit adds up to less and less of what a steady one would.

7 · A differential equation: a rule that gives the rate from the state

EXERCISE 1. Take the symbols from left to right.

- ΔEIi: the change in energy intake from the start, in the interval i. Kilocalories a day.
- ρ: rho, "the effective energy density associated with the body weight change". Kilocalories per kilogram.
- dBWi/dt: the rate of change of body weight over the interval, worked out from the moving average of the weighings. Kilograms a day.
- ε: epsilon, which "defined how energy expenditure depends on body weight". Kilocalories a day for each kilogram.
- BWi − BW0: the weight in the interval minus the weight at the start. Kilograms.
- Δδ: the change in physical activity. The paper says it is "often assumed to be zero" when there are no measurements of activity.
- β: beta, which "accounts for the adaptation of energy expenditure during a diet perturbation". Table 1 gives it as 0.24, with no unit.
- UGE: urinary glucose excretion, "the energy losses as a result of increased urinary excretion of glucose with canagliflozin treatment". Table 1 gives it in kcal/d, so kilocalories a day.

Check the units of the first two terms on the right. Kilocalories per kilogram times kilograms a day is kilocalories a day. Kilocalories a day per kilogram times kilograms is kilocalories a day. Both match the left side.

This section set the activity term and the glucose term to zero. Setting Δδ to zero assumes physical activity did not change. Setting UGE to zero assumes no extra glucose is lost in the urine, as for a person not taking a drug that raises it. What remains, rearranged, is ρ × dW/dt = ΔEI − ε × (W − W0).

EXERCISE 2. Start from what they already accept. Write the identity in rate form.

```
ρ × dW/dt = EI - EE
```

It is true whenever ρ is fixed, and it cannot predict tomorrow. Tomorrow's EE is not known today.

Then add the one piece of physiology. A smaller body spends less. Draw expenditure as a straight line rising with weight, slope ε. Rewrite the right side as the cut in intake minus ε times the change in weight.

```
ρ × dW/dt = ΔEI - ε × (W - W0)
```

Now point at the second term. On day one it is zero, so the whole cut is the deficit. As W falls below W0, the term takes a bite out of the deficit. The same cut buys a smaller deficit, so a slower loss.

Put numbers in if they want them. Hall's 100 kg man cuts 480 kilocalories a day. With ε at 20, once he is 10 kilograms down the fall in expenditure is 200 kilocalories a day. His deficit is then 280, not 480.

Close with what the equation does not say. It does not say ε is 20 for this patient. It does not include appetite rising as weight falls. And it is a straight-line simplification.

1. Put each value of y into the right side.

```
0.5 times 4 = 2
6 minus 2 = 4
0.5 times 12 = 6
6 minus 6 = 0
0.5 times 20 = 10
6 minus 10 = -4
```

The rates are 4, 0 and minus 4. The rate depends on where y is. Below 12, y rises. Above 12, y falls.

2. A differential equation holds an unknown function and its derivative.

1. is not one. It gives y directly and holds no derivative.
2. is one. Its rate depends on t alone. An antiderivative of the right side solves it, which is the earlier section's move.
3. is one. Its rate depends on the state y. You cannot integrate the right side without already knowing y at every time, which is what you are looking for.
4. is not one. It is a weighted sum of three quantities, with no rate in it.

Equations like (c) are the ones this section is about.

3. Each step: rate from the current y, then new y = old y + 1 × rate.

```
6 minus 0.5 times 4 = 4
4 plus 4 = 8
6 minus 0.5 times 8 = 2
8 plus 2 = 10
6 minus 0.5 times 10 = 1
10 plus 1 = 11
```

About 8 at t = 1, 10 at t = 2 and 11 at t = 3. Each step is smaller than the one before, because the rate falls as y rises.

4. The start first. At t = 0, $e^{(0)}$ is 1.

```
12 minus 8 = 4
```

The initial value holds.

The left side. With k = 0.5, the derivative of $e^{(-kt)}$ is $-k \times e^{(-kt)}$.

```
dy/dt = -8 × (-0.5) × e^(-0.5t) = 4 × e^(-0.5t)
```

The right side.

```
6 - 0.5 × (12 - 8 × e^(-0.5t)) = 6 - 6 + 4 × e^(-0.5t) = 4 × e^(-0.5t)
```

The two sides match at every t, so the function solves the problem.

At t = 2 the exponent is minus 1, and $e^{(-1)}$ is 0.3679.

```
8 times 0.3679 = 2.9432
12 minus 2.9432 = 9.0568
```

About 9.06. Forward steps of size 1 give 10 at t = 2. The steps overshoot, because each holds the first rate of its step for the whole step while the true rate is falling.

5. Check each at t = 0, then on both sides.

1. At t = 0, 12 + 8 = 20. The start holds. The left side is $-4 \times e^{(-0.5t)}$. The right side is $6 - 6 - 4 \times e^{(-0.5t)}$, the same. (a) solves it.
2. At t = 0, y = 20. The start holds. The left side is minus 4 at every t. The right side is $6 - 0.5 \times (20 - 4t)$, which is $-4 + 2t$. The two agree only at t = 0. At t = 1:

```
20 minus 4 = 16
0.5 times 16 = 8
6 minus 8 = -2
```

Minus 2 on the right against minus 4 on the left. (b) is not a solution.

1. The left side is $-2 \times e^{(-0.5t)}$. The right side is $6 - 6 - 2 \times e^{(-0.5t)}$, the same, so it obeys the rule. But at t = 0 it gives 12 + 4 = 16, not 20. It is a solution of the equation and the wrong member of the family.

6. On day one, W equals W0, so the ε term is zero. The equation is 7,716.18 × dW/dt = −300.

```
300 divided by 7716.18 = 0.03888
0.03888 times 7 = 0.27216
```

dW/dt is about minus 0.0389 kilograms a day, or minus 0.272 kilograms a week.

Check the unit. Kilocalories a day divided by kilocalories per kilogram leaves kilograms a day.

This rate holds on day one only. Every later day, the weight is lower and the rate is smaller.

7. The equation is 7,716.18 × dW/dt = −300 − ε × (W − W0), with W − W0 = −4.

With ε = 20:

```
20 times 4 = 80
300 minus 80 = 220
220 divided by 7716.18 = 0.02851
```

With ε = 30:

```
30 times 4 = 120
300 minus 120 = 180
180 divided by 7716.18 = 0.02333
```

About minus 0.0285 and minus 0.0233 kilograms a day. On day one it was minus 0.0389. Four kilograms in, the loss has slowed by a quarter to two-fifths, depending on ε. The range of ε is carried into the answer. It is not a rounding error to be smoothed away.

8. The rate first, from the two weighings.

```
95.8 minus 96.4 = -0.6
0.6 divided by 52 = 0.011538
```

dW/dt is about minus 0.011538 kilograms a day. Now the change since the start.

```
98.0 minus 95.8 = 2.2
```

W − W0 is minus 2.2 kilograms. Rearrange the equation for ΔEI by adding ε × (W − W0) to both sides.

```
ΔEI = ρ × dW/dt + ε × (W - W0)
7716.18 times 0.011538 = 89.03
30 times 2.2 = 66
89.03 plus 66 = 155.03
```

ΔEI is about minus 155 kilocalories a day. Both terms are negative here. The weight is still falling, and it is below the start.

9. The right side is 0 − ε × (−5), which is plus 5 × ε.

With ε = 20:

```
20 times 5 = 100
100 divided by 7716.18 = 0.012960
0.012960 times 30 = 0.3888
```

With ε = 30:

```
30 times 5 = 150
150 divided by 7716.18 = 0.019440
0.019440 times 30 = 0.5832
```

dW/dt is plus 0.013 to plus 0.019 kilograms a day. Over 30 days that is about 0.39 to 0.58 kilograms, if the rate held. It will not quite hold: as the weight rises the bracket shrinks and the rate falls. The weight is heading back up.

The paper describes this course as a plateau “followed by slow weight regain”. The equation gives the same direction, from intake alone and the lower expenditure of a smaller body. Nobody’s effort appears anywhere in it.

10. Step 1, from day 0. The bracket is zero.

```
480 divided by 7716.18 = 0.062207
0.062207 times 30 = 1.86621
100 minus 1.86621 = 98.13379
```

Step 2, from 98.13379. The weight is 1.86621 below the start.

```
20 times 1.86621 = 37.3242
480 minus 37.3242 = 442.6758
442.6758 divided by 7716.18 = 0.057370
0.057370 times 30 = 1.7211
98.13379 minus 1.7211 = 96.41269
```

Step 3, from 96.41269. The weight is 3.58731 below the start.

```
20 times 3.58731 = 71.7462
480 minus 71.7462 = 408.2538
408.2538 divided by 7716.18 = 0.052909
0.052909 times 30 = 1.58727
96.41269 minus 1.58727 = 94.82542
```

About 94.83 kilograms at day 90. The static rule:

```
480 times 90 = 43200
43200 divided by 7716.18 = 5.5986
100 minus 5.5986 = 94.4014
```

About 94.40 kilograms. The two differ by about 0.4 kilograms at 90 days, and the gap widens every month after. The curve formula from the illustration gives 95.01 at day 90. Thirty-day steps overshoot the loss a little, 94.83 against 95.01, because each holds its first rate for 30 days.

11. The step that broke is the third. It takes the rate from day 0 and uses it for all 90 days. The equation gives a new rate every day from that day’s weight. By day 30 the weight is lower, so the ε term is no longer zero, and the rate is smaller.

Stepping properly, as in the problem above with 30-day steps, gives about 94.83 kilograms. The curve formula gives 95.01.

Why the wrong answer looks reasonable. Lines one and two are correct. The day-0 rate is right. The error is to treat a rate at an instant as a rate for the whole stretch. That turns the model back into the static rule, which is exactly what 94.40 is.

12. The step that broke is the second: the sign in front of ε. The model should read $\rho \times dW/dt = \Delta EI - \varepsilon \times (W - W0)$.

Test it with the physiology. When W is below W0, the body is smaller, so expenditure is lower. Lower expenditure means less going out, so the deficit should shrink, not grow. With the minus sign:

```
20 times 4 = 80
300 minus 80 = 220
220 divided by 7716.18 = 0.02851
```

The rate is minus 0.0285 kilograms a day, slower than the day-one 0.0389, as it should be.

Why the wrong answer looks reasonable. Every line after the second is correct arithmetic. A plus and a minus look alike in a methods section. The quick check is the direction: a model in which losing weight makes you lose faster runs away, and nobody's weight does that.

13. The step that broke is the third. It divides megajoules by kilocalories per kilogram. The units do not cancel to kilograms.

Convert the cut first. The paper gives it as 480 kilocalories a day.

```
480 divided by 7716.18 = 0.062207
```

Minus 0.0622 kilograms a day on day one, about 240 times the wrong figure.

Why the wrong answer looks reasonable. Every line is correct arithmetic, and a small number after a division feels plausible. A unit check on line three catches it: megajoules a day divided by kilocalories per kilogram is not kilograms a day. So does a size check. A cut of 480 kilocalories a day moving weight by 95 grams in a year should stop anyone.

14. The step that broke is the second. W0 is the weight at the start of the study, 98.0 kilograms, not the weight at the previous weighing. The ε term is about how far expenditure has fallen since the start. That depends on the whole loss so far, not on the last 52 days.

```
98.0 minus 95.8 = 2.2
30 times 2.2 = 66
89.03 plus 66 = 155.03
```

ΔEI is about minus 155 kilocalories a day, not minus 107.

Why the wrong answer looks reasonable. The first line correctly uses the two latest weighings for the rate. Using the same two weighings for the change looks consistent. It mixes up two quantities: the rate is recent, and the change is since the start.

15. What to compute: whether the prediction comes from the identity or from something added to it. Take Hall's man, cutting 480 kilocalories a day, with ρ = 7,716.18. Work out his rate when he is 4 kilograms down, under three rules for expenditure.

ε = 0, expenditure ignores weight:

```
480 divided by 7716.18 = 0.06221
```

ε = 20:

```
20 times 4 = 80
480 minus 80 = 400
400 divided by 7716.18 = 0.05184
```

ε = 30:

```
30 times 4 = 120
480 minus 120 = 360
360 divided by 7716.18 = 0.04666
```

Three different rates, all obeying the same identity. So the commentator is half right. The identity alone predicts nothing. The prediction comes from the rule for expenditure that was added, and that rule is a claim about bodies, open to test. Nothing about it is circular.

What this does not establish. It does not show which ε is right, or that one straight line fits expenditure across a large loss. It does not show that ρ is constant; Chow and Hall say it need not be. And it says nothing about why intake or expenditure changed. Hall and Guo make that point about the identity itself.

16. What to compute: whether the number depends on anything besides the weighings. Use the made-up person from problem 8: dW/dt = −0.011538 kilograms a day and W − W0 = −2.2 kilograms. Compute ΔEI = ρ × dW/dt + ε × (W − W0) three ways.

ρ = 7,716.18 and ε = 30, as before:

```
7716.18 times 0.011538 = 89.03
30 times 2.2 = 66
89.03 plus 66 = 155.03
```

ρ = 7,716.18 and ε = 20:

```
20 times 2.2 = 44
89.03 plus 44 = 133.03
```

ρ taken as the energy of body fat, 39.5 megajoules a kilogram (Book 1), which is 9,440.73 kilocalories, with ε = 30:

```
39500 divided by 4.184 = 9440.73
9440.73 times 0.011538 = 108.93
108.93 plus 66 = 174.93
```

The same two weighings give minus 133, minus 155 or minus 175 kilocalories a day. So the intake figure is calculated through a model, and it moves with the model's constants. The paper's own word is "calculated".

What this does not establish. It does not show that Polidori's figures are wrong. Their ρ and ε come from a model built on controlled feeding studies, not from these stand-ins. It does not tell you any one person's intake; the paper "restricted our analysis to the group mean". And it assumes activity did not change. The paper puts the bias from that at "<40 kcal/day when studying >100 individuals". Say "calculated", and name the model.

8 · Equilibrium: where the rate is zero

EXERCISE 1. What is wrong: the note runs the static rule, which has no equilibrium when intake changes. It predicts the same loss every year for ever.

Put in its place an equilibrium. Hall and colleagues (2011) give a rule of thumb for an average overweight adult. Every 100 kilojoules a day of change in intake leads to "an eventual bodyweight change of about 1 kg". "Eventual" is the equilibrium. Convert the cut to kilojoules with 4.184 kilojoules per kilocalorie.

```
100 times 4.184 = 418.4
418.4 divided by 100 = 4.184
```

About 4.2 kilograms, once, eventually, not every year. The same paper says half the change comes in about 1 year and 95% in about 3 years.

What the replacement still does not say. The rule of thumb leaves out intake rising as weight falls, which would shrink the figure. It is for an average overweight adult, not a population with a spread of weights. And a cut "sustained" across a population is itself an assumption, not a measurement.

EXERCISE 2. Start with a picture. Two lines on a graph. One is what the body spends, and it rises as the body gets heavier. The other is what the person eats. Where they cross, the weight stays put.

Then the diet. Eating less lowers the eating line. The crossing moves to a lower weight, and the body drifts down to it over months to years.

Then the part people miss. When the diet ends and eating goes back to where it was, the crossing goes back to where it was. The body drifts back up to it. Nobody has to lose discipline for that to happen. It is where the lines cross.

Then the second push. Appetite rises as weight falls. The first study to measure it, in 2016, in people living normally, found about 100 kilocalories a day for each kilogram lost. That tilts the eating line and pulls the crossing back toward the old weight.

The quotable sentence: “A body settles where what it eats meets what it spends. Go back to the old eating and you go back to the old weight. That is physics and physiology, not weakness.”

What not to say: that everyone regains, or by how much. The figures are group averages, and people differ widely.

1. Set the rate to zero and solve.

```
0 = 12 - 0.4 × y
0.4 × y = 12
12 divided by 0.4 = 30
```

The equilibrium is y = 30. Check: 0.4 × 30 is 12, and 12 − 12 is 0.

2. The rate is zero when y − 70 is zero, so the equilibrium is y = 70.

```
60 minus 70 = -10
-0.05 times -10 = 0.5
80 minus 70 = 10
-0.05 times 10 = -0.5
```

At 60 the rate is plus 0.5, so y rises toward 70. At 80 it is minus 0.5, so y falls toward 70. From either side the state returns. The equilibrium is stable.

3. At the crossing, intake equals expenditure.

```
2000 = 500 + 25 × W
2000 minus 500 = 1500
1500 divided by 25 = 60
```

W = 60. Now with intake at 1,800.

```
1800 minus 500 = 1300
1300 divided by 25 = 52
60 minus 52 = 8
```

The crossing moves from 60 to 52, a shift of 8. Check it the short way: the drop in intake divided by the slope.

```
200 divided by 25 = 8
```

4. Set the two equal and gather the W terms on one side by adding 60 × W to both sides.

```
5700 - 60 × W = 500 + 20 × W
5700 minus 500 = 5200
60 plus 20 = 80
5200 divided by 80 = 65
```

W = 65. With intake 400 lower, the left side starts at 5,300.

```
5300 minus 500 = 4800
4800 divided by 80 = 60
```

W = 60, a shift of 5. The short way uses the two slopes added.

```
400 divided by 80 = 5
```

Had intake been flat, the shift would have been 400 divided by 20, which is 20. The sloping intake line cut it to a quarter.

5. The gap y − 50 is multiplied by $e^{(-0.02t)}$. The time constant is 1 divided by 0.02.

```
1 divided by 0.02 = 50
50 times 0.6931 = 34.655
```

The time constant is 50 and the half-time about 34.7. At t = 50 the gap has been multiplied by $e^{(-1)}$.

```
10 times 0.3679 = 3.679
50 plus 3.679 = 53.679
```

y is about 53.68. After one time constant, a little over a third of the gap is left, not none of it.

6. The shift is the change in intake divided by the slope. Rearrange for the slope by multiplying both sides by the slope and dividing both by the shift.

```
slope = change in intake ÷ shift
300 divided by 12 = 25
```

A slope of 25. Check: 300 divided by 25 is 12.

7. Intake is flat, so the shift is the cut divided by the slope.

```
300 divided by 20 = 15
300 divided by 30 = 10
```

The equilibrium falls by 10 to 15 kilograms. Check the unit: kilocalories a day, divided by kilocalories a day per kilogram, leaves kilograms.

The range comes from the range of slopes. Quote both ends, not the middle.

8. Intake rises as weight falls, and expenditure falls as weight falls. Both push the weight back, so the slopes add.

```
100 plus 20 = 120
300 divided by 120 = 2.5
100 plus 30 = 130
300 divided by 130 = 2.307692308
```

About 2.3 to 2.5 kilograms. The settling point model gave 10 to 15.

```
15 divided by 2.5 = 6
10 divided by 2.307692308 = 4.333333333
```

The set point shift is about a sixth to a quarter of the settling point shift, depending on the slope of expenditure.

9. The first form, 100 kilojoules a day per kilogram.

```
100 divided by 4.184 = 23.90
```

The second form, 10 kilocalories a day per pound. There are 1 divided by 0.45359237 pounds in a kilogram, so divide by 0.45359237.

```
10 divided by 0.45359237 = 22.05
```

About 23.90 and 22.05 kilocalories a day per kilogram. Both sit inside Hall and Guo's 20 to 30. The shifts:

```
300 divided by 23.90 = 12.55
300 divided by 22.05 = 13.61
```

About 12.6 and 13.6 kilograms. The paper's "equivalently" is a rounding, and the two forms differ by about 8%. That is fine for a rule of thumb. Quote which form you used.

10. Equation 4 is ΔEI = −kP × ΔW. Divide both sides by minus kP.

```
ΔW = -ΔEI ÷ kP
350 divided by 100 = 3.5
```

ΔW is about minus 3.5 kilograms. The paper says the new equilibrium was "several kilograms lower". The sizes agree.

Both figures are rounded ("~350", about 100), so this checks that the paper's numbers fit together. It does not replace the paper's own weight curve.

11. The shift is the cut divided by the slope.

```
480 divided by 23.90 = 20.08
100 minus 20.08 = 79.92
```

About 80 kilograms by the rule of thumb, against about 75 from the paper's model. The model's loss is about 25 kilograms, the rule's about 20.

The rule of thumb is for "an average overweight adult". If he carries more body fat than the average overweight adult the rule was built for, the paper says his loss will be larger. Then the rule's single slope is too steep for him, and its equilibrium comes out too high.

A single slope is the simplification. The model lets the slope depend on the body.

12. The step that broke is the second. Both feedbacks push the weight back toward where it was, so their slopes add.

```
100 plus 20 = 120
300 divided by 120 = 2.5
```

The shift is 2.5 kilograms, not 3.75.

Why the wrong answer looks reasonable. The intake slope is written with a minus sign and the expenditure slope without one. Subtracting feels like respecting the signs. Check the direction instead. As weight falls, intake goes up and expenditure goes down. Both close the gap, so both count toward the total slope.

13. The step that broke is the second. The time constant is ρ divided by the slope, not the slope divided by ρ.

```
7716.18 divided by 20 = 385.809
```

About 386 days.

A unit check finds it. Kilocalories per kilogram divided by kilocalories a day per kilogram leaves days. The inverted ratio has the unit one over days: it is a rate, 0.00259 of the gap closed per day, not a time. Reading it as days turns "about a year" into "under four minutes".

Why the wrong answer looks reasonable. Both numbers are the right ones, and dividing them is the right operation. Only the order is wrong, and a small answer then looks like a quick settling.

14. The step that broke is the third. An equilibrium is a weight at which the rate is zero and stays zero, because nothing in the rule is changing. At the lowest point the rate was zero for a moment only. Intake was still climbing back toward where it began. The paper describes intake relaxing toward baseline, then "slow weight regain".

Check with the settling point model. A year in, the paper has intake "practically at baseline levels while body weight was still reduced by ~5 kg". With ΔEI = 0 the equilibrium is the starting weight. At about 5 kilograms below it, with ε = 20:

```
20 times 5 = 100
```

The right side is plus 100 kilocalories a day, so the weight is rising.

Why the wrong answer looks reasonable. A zero rate is the test for an equilibrium, and the weight did stop falling. The missing check is whether the rule itself had stopped moving. A ball thrown upward also has zero speed at the top, and it does not stay there.

15. The step that broke is the second. After one time constant the gap has been multiplied by $e^{(-1)}$, which is 0.3679. About 37% of it is still left.

```
1 minus 0.3679 = 0.6321
24 times 0.6321 = 15.17
100 minus 15.17 = 84.83
```

On day 386 he weighs about 84.8 kilograms, not 76. In this model he never quite reaches 76. The gap keeps shrinking by the same fraction.

Why the wrong answer looks reasonable. "Time constant" sounds like a time to arrive. It is the time over which the gap shrinks by one factor of e. For a date you can use, give the time to 95% of the shift: τ times ln 20, about 1,156 days here.

16. What to compute: which model gives 15 kilograms, over what time, and what the other model in the same paper gives.

The 15 is the settling point model's shift at the low end of its slope, from Hall and Guo.

```
300 divided by 20 = 15
300 divided by 30 = 10
```

So 10 to 15 kilograms, eventually. Hall and Guo say it "takes years to equilibrate". With the stand-in ρ of 7,716.18 and a slope of 20, 95% of the shift takes about 1,156 days.

The set point model, in the same paragraph of the same paper, adds appetite rising about 100 kilocalories a day per kilogram lost.

```
300 divided by 120 = 2.5
300 divided by 130 = 2.307692308
```

About 2.3 to 2.5 kilograms.

What this does not establish. It does not say which model fits a given reader; Hall and Guo expect "a wide degree of individual variation". It does not say the cut can be kept up for years. It does not say the lines stay straight over a loss of 15 kilograms. The paper that measured the appetite slope does not yet know that. A fair headline would say "somewhere between about 2 and 15 kg, over years, depending on how much appetite rises".

17. What to compute: whether an equilibrium in a set point model is fixed, or moves when intake is shifted. Use Hall and Guo's slopes: intake about −100 kilocalories a day per kilogram, expenditure 20 to 30. Take a further sustained shift of 200 kilocalories a day in her intake line.

```
200 divided by 120 = 1.666666667
200 divided by 130 = 1.538461538
```

About 1.5 to 1.7 kilograms lower, eventually. Small, and not zero. In Hall and Guo's set point model the equilibrium moves whenever the intake line moves. "Set point" names the model, in which both flows push back. It does not name a weight that cannot change.

What this does not establish. It does not show that two flat months are an equilibrium. Water shifts alone can hide a slow trend over the short term, and a flat stretch can be a turning point while intake is still moving. It does not give her own slopes, which may differ widely from the averages. It does not show that a 200 kilocalorie shift is something she can sustain, or what it would cost her in hunger. And the straight lines may not hold over her range; Polidori and colleagues say so of their own. So answer the colleague with "it moves less than you'd hope, not never", and name the model.

9 · Highest and lowest points: slope zero and curvature

EXERCISE 1. Line 1 is the definition: the misses from m, squared and added over every reading.

Line 2 opens each square into $x^2 - 2mx + m^2$. It multiplies every term of one bracket by every term of the other. It then adds over the n readings. The candidate m is the same in every term, so it comes outside the sum, and m^2 is added n times.

Line 3 differentiates with respect to m, term by term. $\Sigma\, x^2$ has no m, so it gives zero. The fixed number $-2\,\Sigma\, x$ times m gives $-2\,\Sigma\, x$. And n m^2 gives 2nm by the power rule.

The missing line sets the derivative to zero and moves one term across.

```
2nm = 2 Σ x
```

Line 4 divides both sides by 2n.

Line 5 differentiates line 3 once more. The first term has no m and gives zero, and 2nm gives 2n. Because 2n is positive, the critical point in line 4 is a minimum.

1. Differentiate.

```
f′(x) = 2x - 6
f″(x) = 2
```

Set the first to zero: $2x = 6$.

```
6 divided by 2 = 3
```

The critical point is $x = 3$. The second derivative is 2, which is positive, so it is a local minimum. Now the value.

```
3 times 3 minus 6 times 3 plus 11 = 2
```

A local minimum of 2, at $x = 3$.

2. Differentiate.

```
g′(x) = -4x + 8
g″(x) = -4
```

Set g' to zero: $4x = 8$.

```
8 divided by 4 = 2
```

The critical point is $x = 2$. The second derivative is -4, less than zero, so it is a local maximum.

```
-2 times 2 times 2 plus 8 times 2 plus 1 = 9
```

A local maximum of 9, at $x = 2$.

3. Differentiate.

```
h′(x) = 3x² - 12
h″(x) = 6x
```

Set h' to zero: $3x^2 = 12$, so $x^2 = 4$. Both 2 times 2 and -2 times -2 are 4, so $x = 2$ or $x = -2$.

At $x = 2$ the second derivative is 12, positive, so a local minimum.

```
2 times 2 times 2 minus 12 times 2 = -16
```

At $x = -2$ the second derivative is -12, negative, so a local maximum.

```
-2 times -2 times -2 minus 12 times -2 = 16
```

A local minimum of -16 at $x = 2$, and a local maximum of 16 at $x = -2$.

4. Differentiate twice.

```
f′(x) = 4x³
f″(x) = 12x²
```

Both are zero at $x = 0$. So $x = 0$ is a critical point, and the second derivative test decides nothing.

Use the sign of f' on each side instead. Try $x = -1$ and $x = 1$.

```
4 times -1 times -1 times -1 = -4
4 times 1 times 1 times 1 = 4
```

The slope is negative before 0 and positive after it. The function falls and then rises, so x = 0 is a local minimum, with f(0) = 0.

5. Find the critical point.

```
f'(x) = 2x - 4
```

It is zero at x = 2, which lies inside the interval. Now work out f there and at both ends.

```
0 times 0 minus 4 times 0 plus 5 = 5
2 times 2 minus 4 times 2 plus 5 = 1
5 times 5 minus 4 times 5 plus 5 = 10
```

Compare the three values: 5 at x = 0, 1 at x = 2, and 10 at x = 5. The overall minimum is 1, at the critical point x = 2. The overall maximum is 10, at the end x = 5, where the slope is not zero.

6. The sum is smallest at the mean.

```
3 plus 7 plus 8 plus 12 plus 5 = 35
35 divided by 5 = 7
```

So m = 7. Find the misses from 7, square them and add.

```
3 minus 7 = -4
7 minus 7 = 0
8 minus 7 = 1
12 minus 7 = 5
5 minus 7 = -2
-4 times -4 plus 0 times 0 plus 1 times 1 plus 5 times 5 plus -2 times -2 = 46
```

The smallest value of the sum is 46. The second derivative is 2n, with n = 5.

```
2 times 5 = 10
```

It is 10, positive, as a minimum needs.

7. The least-squares value is the mean.

```
88.4 plus 89.0 plus 88.7 = 266.1
266.1 divided by 3 = 88.7
```

The mean is 88.7 cm. The misses from it are −0.3, 0.3 and 0 cm.

```
-0.3 times -0.3 plus 0.3 times 0.3 plus 0 times 0 = 0.18
```

The smallest sum is 0.18 square centimetres, cm^2.

Reporting 89.0 cm is a step of 0.3 cm off the mean. The sum rises by n times the square of the step, with n = 3.

```
3 times 0.3 times 0.3 = 0.27
0.18 plus 0.27 = 0.45
```

Check it directly. The misses from 89.0 are −0.6, 0 and −0.3 cm.

```
-0.6 times -0.6 plus 0 times 0 plus -0.3 times -0.3 = 0.45
```

Both routes give 0.45 cm².

8. Let x be the length, in metres, of each of the two sides that meet the wall. The fence covers those two sides and the side opposite the wall, so that side is 400 − 2x metres long. The area A, in square metres, is one side times the other.

```
A(x) = x times (400 - 2x)
A(x) = 400x - 2x²
```

Differentiate and set to zero.

```
A′(x) = 400 - 4x
A″(x) = -4
```

A′ is zero when 4x = 400.

```
400 divided by 4 = 100
400 minus 2 times 100 = 200
100 times 200 = 20000
```

The second derivative is −4, negative, so x = 100 is a maximum. x can run from 0 to 200, and at both ends the area is zero, so this is the overall maximum. The ground should be 100 m out from the wall and 200 m along it, for an area of 20,000 square metres.

9. The sum of squared misses is a quadratic in the reported value m. Its lowest point is at the mean, and its second derivative is 2n. A step h from the lowest point raises it by half of 2n times h², which is n times h². Here the step is 0.5 kg.

```
70.5 minus 70.0 = 0.5
4 times 0.5 times 0.5 = 1
16 times 0.5 times 0.5 = 4
```

A's sum rises by 1 kg², and B's by 4 kg².

The same step off the mean costs four times as much for B, because B has four times as many readings. B's valley is sharper. The more readings there are, the more firmly they pin the best single value down.

10. Compare with the general form S(m) = Σ x² − 2m Σ x + n m². The number in front of m² is n, so there were 4 readings. The number in front of m is −2 Σ x, so Σ x is 280.

```
560 divided by 2 = 280
280 divided by 4 = 70
```

The mean is 70 kg. Check it by the derivative instead.

```
S′(m) = 8m - 560
560 divided by 8 = 70
```

The two routes agree. Now S at the mean.

```
4 times 70 times 70 minus 560 times 70 plus 19,601 = 1
```

The smallest sum of squared deviations is 1 kg^2. Divide it by one less than the count.

```
1 divided by 3 = 0.333…
```

The sample variance is about 0.333 kg^2.

11. The last step broke. The working found a critical point and never asked which kind it was.

```
f″(x) = -2
```

The second derivative is negative, so x = 3 is a maximum, not a minimum. The largest value f takes is 9. It has no smallest value: as x moves far from 3 in either direction, $-x^2$ takes over and f falls without end.

Why the wrong answer looks reasonable: the first three lines are all correct. Setting the derivative to zero is the right first move, and it produces a tidy number. The error is the word “smallest”, which the working assumed and never checked.

12. The last step broke. Everything before it is right, but it compared only the critical points and left out the two ends of the interval.

```
2 times -1 times -1 times -1 minus 3 times -1 times -1 = -5
2 times 2 times 2 times 2 minus 3 times 2 times 2 = 4
```

f(−1) is −5 and f(2) is 4. Set these beside the critical values of 0 and −1. The overall maximum is 4, at the end x = 2. The overall minimum is −5, at the end x = −1. Both overall answers sit at ends, and neither is at a critical point.

Why the wrong answer looks reasonable: each critical point was found and sorted correctly, and “local maximum” sounds like “the maximum”. The second derivative test only compares a point with its near neighbours. It says nothing about the ends.

13. The second line broke. When the square is added up over all n readings, m^2 appears once in every term, so it is added n times. The line should read as follows.

```
S(m) = Σ x² - 2m Σ x + n m²
```

Then S′(m) = −2 Σ x + 2nm, and setting it to zero gives m = Σ x divided by n, the mean.

The wrong answer, m = Σ x, fails a simple check. For the readings 3, 7, 8, 12 and 5 it gives m = 35. That is larger than every reading, which no centre of those readings can be.

Why it looks reasonable: the m came out of the sum correctly in the middle term. It is easy to treat m^2 the same way without counting how many times it was added.

14. The third step broke. A rate of change of zero at one instant marks a turning point. An equilibrium, in the sense of section 8, is different. It is a weight at which the rule that gives the rate gives zero and keeps giving zero, so the weight stays.

Her weight did not stay at 82 kg. The first line says it fell afterwards. So at week 6 the rate was zero only for an instant. That happens at the top of any smooth curve that turns. 82 kg is the highest point of her weight over those weeks. It is not an equilibrium.

Why it looks reasonable: both an equilibrium and a maximum are found by setting a derivative to zero. The words “rate of change is zero” fit both. What separates them is whether the zero lasts.

15. What to compute: whether the loss had levelled off by the top dose, which is what a peak inside the tested doses would need. Work out the average rate of change between neighbouring doses.

```
3.5 minus 2.0 = 1.5
1.5 divided by 10 = 0.15
4.5 minus 3.5 = 1.0
1.0 divided by 10 = 0.1
```

Between 10 and 20 units, loss rose by 0.15 kg per unit. Between 20 and 30, it rose by 0.1 kg per unit. The slope is smaller but still positive at the top of the tested doses.

So 30 units gave the most loss among the doses tested. That largest value sits at an end of the tested doses, where the slope has not reached zero. It is not a peak found inside them.

What this does not establish. It does not show that 30 units is the best dose. A higher dose might give more loss, since the slope had not reached zero, or it might give less. Three points cannot locate a peak that lies outside them. "Optimal" also needs a statement of what is being made best. The table measures loss only. It says nothing about side effects, cost, or how long the loss lasts.

16. What to compute: best by what measure. The mean is best when the misses are squared and added. Compare the mean with the median by that measure and by the plain sizes of the misses.

```
70.2 plus 70.4 plus 70.3 plus 70.5 plus 72.6 = 354
354 divided by 5 = 70.8
```

The mean is 70.8 kg. In order, the readings are 70.2, 70.3, 70.4, 70.5 and 72.6, so the median is 70.4 kg.

Squared misses from each.

```
-0.6 times -0.6 plus -0.4 times -0.4 plus -0.5 times -0.5 plus -0.3 times -0.3 plus
1.8 times 1.8 = 4.10
-0.2 times -0.2 plus 0 times 0 plus -0.1 times -0.1 plus 0.1 times 0.1 plus 2.2
times 2.2 = 4.90
```

By squared misses the mean wins, 4.10 kg^2 against 4.90. Now the sizes of the misses, not squared.

```
0.6 plus 0.4 plus 0.5 plus 0.3 plus 1.8 = 3.6
0.2 plus 0 plus 0.1 plus 0.1 plus 2.2 = 2.6
```

By plain sizes the median wins, 2.6 kg against 3.6. The mean is best only under the squared measure. Squaring made the one reading in outdoor clothes count for most of the total, and it pulled the mean above four of the five readings.

What this does not establish. It does not say which number is the person's true weight. Neither calculation knows that the last reading was taken in clothes. Dropping it or keeping it is a decision about the measurement, made before any arithmetic. It also does not say the median is always better. With no far-off reading, the two sit close together.

10 · Vectors as data rows, and the weighted sum

EXERCISE 1. In words first. The energy is 17 times the grams of protein, plus 37 times the grams of fat, plus 17 times the grams of carbohydrate.

```
E = 17p + 37f + 17c
```

E is the food's energy in kJ. The letters p, f and c are the grams of protein, fat and carbohydrate, each in g. The numbers 17, 37 and 17 are in kJ/g, so each product is in kJ, and the three can be added.

As a dot product, name the factors k = (17, 37, 17) and the grams g = (p, f, c), in the same order.

```
E = k · g
```

The vector k holds the weights. Each entry of k is the change in E, in kJ, for one more gram of its nutrient with the other two held fixed.

1. Add entry by entry.

```
2 plus 3 = 5
5 plus -1 = 4
1 plus 4 = 5
```

u + v = (5, 4, 5). Now multiply every entry of u by 3.

```
3 times 2 = 6
3 times 5 = 15
3 times 1 = 3
```

3u = (6, 15, 3).

2. Scale each vector first.

```
2u = (4, 10, 2)
3v = (9, -3, 12)
```

Then subtract entry by entry.

```
4 minus 9 = -5
10 minus -3 = 13
2 minus 12 = -10
```

2u − 3v = (−5, 13, −10).

3. Multiply matching entries and add.

```
2 times 3 plus 5 times -1 plus 1 times 4 = 5
```

u · v = 5.

```
4 times 0 plus 9 times 1 plus 4 times 0 = 9
4 times 1 plus 9 times 1 plus 4 times 1 = 17
```

Dotting with (0, 1, 0) picks out the middle weight, 9. Dotting with (1, 1, 1) adds all the weights, 17.

4. 1. Cannot. A dot product needs two vectors of the same length, and these have three entries and two. 2. Cannot. Vectors of different lengths cannot be added. 3. Can. Any vector can be multiplied by a scalar.

```
2.5 times 4 = 10
2.5 times 0 = 0
2.5 times -2 = -5
```

The result is (10, 0, −5).

1. Can. Both have length three.

```
1 times 4 plus 2 times 5 plus 3 times 6 = 32
```

The dot product is 32.

5. Write the grams as g = (7, 18, 66) and the factors as a = (4, 9, 4), in the same order. The energy is a · g.

```
4 times 7 = 28
9 times 18 = 162
4 times 66 = 264
28 plus 162 plus 264 = 454
```

100 g gives 454 kcal.

6. Dot the grams with (17, 37, 17).

```
17 times 7 = 119
37 times 18 = 666
17 times 66 = 1122
119 plus 666 plus 1122 = 1907
```

100 g gives 1,907 kJ.

```
1907 divided by 454 = 4.200…
```

The ratio is about 4.2. A kilocalorie is about 4.18 kJ (Book 0, B4), and the rounded factors (17 for 4.0, 37 for 9.0) move the ratio a little off that.

7. A 25 g serving is 0.25 of 100 g. Scale the gram vector.

```
0.25 times 7 = 1.75
0.25 times 18 = 4.5
0.25 times 66 = 16.5
```

The serving holds (1.75, 4.5, 16.5) g. Dot with (4, 9, 4).

```
4 times 1.75 plus 9 times 4.5 plus 4 times 16.5 = 113.5
0.25 times 454 = 113.5
```

The serving gives 113.5 kcal. Scaling the grams by 0.25 scaled the energy by 0.25.

8. Choose an order and keep it for both vectors: protein, fat, carbohydrate, alcohol.

```
g = (0, 0, 10, 14)
a = (4, 9, 4, 7)
4 times 0 plus 9 times 0 plus 4 times 10 plus 7 times 14 = 138
```

The drink gives 138 kcal, and 98 of them come from the alcohol.

9. The energy is 4 times protein plus 9 times fat plus 4 times carbohydrate. Two of the three products are known.

```
4 times 8 = 32
9 times 24 = 216
480 minus 32 minus 216 = 232
```

So 4 times the carbohydrate is 232 kcal.

```
232 divided by 4 = 58
```

The label should read 58 g of carbohydrate. Check it.

```
4 times 8 plus 9 times 24 plus 4 times 58 = 480
```

10. First way: add the three vectors, then dot once.

```
4 plus 9 plus 1 = 14
2 plus 6 plus 8 = 16
30 plus 12 plus 22 = 64
4 times 14 plus 9 times 16 plus 4 times 64 = 456
```

Second way: dot each food, then add the energies.

```
4 times 4 plus 9 times 2 plus 4 times 30 = 154
4 times 9 plus 9 times 6 plus 4 times 12 = 138
4 times 1 plus 9 times 8 plus 4 times 22 = 164
154 plus 138 plus 164 = 456
```

Both give 456 kcal.

11. The second step broke. The grams were copied in the label's order, fat first. The factors are in the order protein, fat, carbohydrate. So the 30 g of fat was multiplied by the protein factor, 4, and the 10 g of protein by the fat factor, 9.

Put the grams in the factors' order: (10, 30, 50).

```
4 times 10 plus 9 times 30 plus 4 times 50 = 510
```

The food gives 510 kcal per 100 g, not 410.

Why the wrong answer looks reasonable: every multiplication and the addition are correct. And 410 kcal per 100 g is a believable figure for a snack. The error is in which number sat in which place, and a dot product has no way to notice.

12. The third step broke. The grams had already been scaled to the serving. Scaling the factors as well applied the 0.3 twice. The factors are kcal per gram, and a gram gives the same energy whatever the size of the serving.

Dot the serving's grams with the unscaled factors.

```
4 times 3 plus 9 times 9 plus 4 times 15 = 153
```

The serving gives 153 kcal. The wrong answer is 0.3 times 0.3 times 510, which is 45.9.

Why it looks reasonable: scaling one vector by 0.3 is right. It is easy to feel that "for a serving" should apply to everything on the line. Scale either the grams or the factors, never both.

13. What to compute: the change in energy from the whole swap. The swap removes 10 g of fat and adds 10 g of sugar, which counts as carbohydrate. In the order protein, fat, carbohydrate, the change vector is (0, −10, 10). Dot it with (4, 9, 4).

```
4 times 0 plus 9 times -10 plus 4 times 10 = -50
```

The energy falls by 50 kcal, not 90. The blog used the weight on fat, which gives the change with everything else held fixed. The swap did not hold carbohydrate fixed. It added 10 g of it, worth 40 kcal.

What this does not establish. It gives the change in the recipe's calculated energy. It does not say how much a person eating it absorbs, since the factors are general ones. It says nothing about how full the food leaves someone, or what they eat later. It does not say that the swap is a good idea at all.

14. What to compute: whether one energy total can come from different gram vectors. Try three made-up snacks, in the order protein, fat, carbohydrate, and dot each with (4, 9, 4).

```
4 times 10 plus 9 times 30 plus 4 times 50 = 510
4 times 5 plus 9 times 30 plus 4 times 55 = 510
4 times 10 plus 9 times 20 plus 4 times 72.5 = 510
```

All three give 510 kcal per 100 g. The first and third differ by 10 g of fat and 22.5 g of carbohydrate. The weighted sum gathered three numbers into one and threw the rest away. So the energy figure alone cannot tell you what is in the food.

What this does not establish. It does not say either snack is better. It does not say the two real snacks differ: they may, or may not. To know, you need the grams on each label, not the total. And "nutritionally the same" would need far more than three macronutrients: fibre, salt, vitamins and more.

11 · A matrix, and what multiplying by one does

EXERCISE 1. n is the number of people. X is the data matrix, n × 2. Each row is one person. The first column is body weight in kg, and the second column is the pure number 1.

b is the vector of coefficients, length 2. Its first entry, 15.057, is a slope in kcal/day per kg. Its second, 692.2, is an intercept in kcal/day. They are the FAO/WHO/UNU Table 5.2 equation for men aged 18 to 30.

y is a vector of length n, in kcal/day, with one predicted BMR for each person.

The claim, in words: each person's predicted BMR is 15.057 times their body weight in kg, plus 692.2 kcal/day. The same two coefficients are used for everyone.

1. A has 2 rows and 3 columns, so it is 2 × 3.

The entry in row 2, column 3 is −1. The entry in row 1, column 2 is 0.

2. A has 3 columns and x has 3 entries, so Ax is defined. It has one entry for each of A's 2 rows. Dot each row with x.

```
2 times 1 plus 0 times 2 plus 1 times 3 = 5
3 times 1 plus 4 times 2 plus -1 times 3 = 8
```

Ax = (5, 8).

3. The columns of A are (2, 3), (0, 4) and (1, −1). The weights are the entries of x: 1, 2 and 3.

```
1 times 2 plus 2 times 0 plus 3 times 1 = 5
1 times 3 plus 2 times 4 plus 3 times -1 = 8
```

The weighted sum 1(2, 3) + 2(0, 4) + 3(1, −1) is (5, 8), the same as before.

4. Multiply every entry by 3.

```
3 times 2 = 6
3 times 0 = 0
3 times 1 = 3
3 times 3 = 9
3 times 4 = 12
3 times -1 = -3
```

3A has first row (6, 0, 3) and second row (9, 12, −3).

5. Both are 2 × 2 times 2 × 2, so both are defined and 2 × 2. The columns of Q are (0, 1) and (1, 0). Dot each row of P with each column of Q.

```
1 times 0 plus 2 times 1 = 2
1 times 1 plus 2 times 0 = 1
3 times 0 plus 4 times 1 = 4
3 times 1 plus 4 times 0 = 3
```

PQ has rows (2, 1) and (4, 3). Now QP. The columns of P are (1, 3) and (2, 4).

```
0 times 1 plus 1 times 3 = 3
0 times 2 plus 1 times 4 = 4
1 times 1 plus 0 times 3 = 1
1 times 2 plus 0 times 4 = 2
```

QP has rows (3, 4) and (1, 2). The two products differ. PQ swapped P's columns, and QP swapped its rows.

6. 1. The first has 2 columns and the second 2 rows, so it exists, and it is 3 × 4. 2. The first has 4 columns and the second 3 rows, so it does not exist. 3. It exists, and it is 2 × 1: a vector of length 2. 4. It exists, and it is 1 × 1: a single number. One row dotted with one column is the dot product of section 10.

7. The matrix has rows (14.818, 486.6) and (8.126, 845.6). Her vector is (55, 1), with the 1 lined up with the intercepts.

```
14.818 times 55 plus 486.6 times 1 = 1301.59
8.126 times 55 plus 845.6 times 1 = 1292.53
```

The answer is (1301.59, 1292.53) kcal/day. The first entry is the predicted BMR of a 55 kg woman by the equation for ages 18 to 30. The second is by the equation for ages 30 to 60.

8. X is 3 × 2, with rows (45, 1), (55, 1) and (70, 1). b has 2 entries, so Xb is defined. It has 3 entries, one for each woman.

```
14.818 times 45 plus 486.6 times 1 = 1153.41
14.818 times 55 plus 486.6 times 1 = 1301.59
14.818 times 70 plus 486.6 times 1 = 1523.86
```

y = (1153.41, 1301.59, 1523.86), in kcal/day.

Table 5.2 does not give the range of body weights its equations were fitted on. If 45 kg lies outside it, the first entry stretches the line past the data: a model assumption, not arithmetic.

9. A scalar multiplies every entry.

```
1.75 times 1670.905 = 2924.08375
1.75 times 1618.78 = 2832.865
```

The result is about (2924.1, 2832.9) kcal/day. It is the energy requirement at a PAL of 1.75 that each equation's BMR implies for a 65 kg man.

10. X is 3 × 2 and D is 2 × 2, so XD exists and is 3 × 2. Its first column is Xb from the problem before last: 1153.41, 1301.59 and 1523.86. The second column is X times (8.126, 845.6).

```
8.126 times 45 plus 845.6 times 1 = 1211.27
8.126 times 55 plus 845.6 times 1 = 1292.53
8.126 times 70 plus 845.6 times 1 = 1414.42
```

XD has rows (1153.41, 1211.27), (1301.59, 1292.53) and (1523.86, 1414.42), in kcal/day.

At 45 kg the equation for ages 30 to 60 predicts more, 1211.27 against 1153.41. At 55 kg and at 70 kg it predicts less. Its line is flatter, with a smaller slope, and it starts higher, with a larger intercept. So the two lines cross somewhere between 45 and 55 kg. The 45 kg comparison carries the caution of the problem before last.

11. The first step broke. The entries of his vector are in the wrong order. The slope, 15.057, has to meet his body weight, and the intercept, 692.2, has to meet the 1. So his vector must be (65, 1), in the order of b.

```
15.057 times 65 plus 692.2 times 1 = 1670.905
```

His predicted BMR is about 1,671 kcal/day.

Why the wrong answer looks reasonable: a dot product of two vectors of length two is correct arithmetic whatever the order. The error shows only when you read the answer back as a quantity. 45,008 kcal/day is more than twelve times the 3,470 kcal/day ICMR-NIN gives for a man doing heavy work. Read every answer back before you use it.

12. The second step broke. It multiplied entries in matching places, which is not a matrix product. Each entry of PQ is a row of P dotted with a column of Q. The columns of Q are (5, 7) and (6, 8).

```
1 times 5 plus 2 times 7 = 19
1 times 6 plus 2 times 8 = 22
3 times 5 plus 4 times 7 = 43
3 times 6 plus 4 times 8 = 50
```

PQ has rows (19, 22) and (43, 50).

Why the wrong answer looks reasonable: adding two matrices does work entry by entry, and so does multiplying a matrix by a scalar. It feels natural for multiplying two matrices to work the same way. It does not.

13. The first step broke. A 10% cut in the BMR multiplies the whole prediction by 0.9. The prediction is the slope times body weight plus the intercept, so both parts must be scaled. The working scaled the slope and left the intercept alone.

Scale the whole row of coefficients, or scale the answer. The two give the same result.

```
0.9 times 692.2 = 622.98
13.5513 times 65 plus 622.98 = 1503.8145
0.9 times 1670.905 = 1503.8145
```

The cut BMR is about 1,504 kcal/day, not 1,573. The working took off only about 98 kcal/day of the 167 kcal/day it meant to.

Why it looks reasonable: the slope is the part that "depends on weight", so it can feel like the part to adjust. But a scalar multiple of a matrix multiplies every entry. Scaling part of a row is a different model, not a percentage cut.

14. What to compute: what "overestimate by 10%" means as a scalar. Take the 65 kg man's prediction from the equation for men aged 18 to 30, 1670.905 kcal/day.

If the prediction is 10% above the actual figure, the prediction is 1.10 times the actual. Then the actual is the prediction divided by 1.10.

```
1670.905 divided by 1.1 = 1519.005
1670.905 divided by 1.12 = 1491.879
```

If instead the excess is 10% of the prediction, the actual is 0.90 times the prediction.

```
0.9 times 1670.905 = 1503.8145
0.88 times 1670.905 = 1470.3964
```

Dividing by 1.10 is the same as multiplying by about 0.909, not 0.90. The two readings differ by about 15 to 21 kcal/day here. The colleague's rule takes the second reading without saying so.

What this does not establish. It does not say which reading ICMR-NIN meant. Its note gives "10 -12%" and does not define it. The note's own adjustment is a different thing: two 5% cuts. It does not give any one person's true BMR. A range of 10 to 12% for a group says nothing about where one person sits, and the equations were never fitted to that person.

15. What to compute: how much the answer can move for reasons the arithmetic cannot see. Take two men who both weigh 65 kg, one aged 29 and one aged 31. The spreadsheet uses the FAO/WHO/UNU equation for each man's age band.

```
15.057 times 65 plus 692.2 times 1 = 1670.905
11.472 times 65 plus 873.1 times 1 = 1618.78
1670.905 minus 1618.78 = 52.125
```

Two years of age, across the line between bands, move the prediction by about 52 kcal/day at the same body weight. Now the size of the bias ICMR-NIN describes, 10 to 12% of the prediction, for the younger man.

```
0.10 times 1670.905 = 167.0905
0.12 times 1670.905 = 200.5086
```

That is roughly 167 to 201 kcal/day.

The multiplication is exact: the spreadsheet does the arithmetic of the equations without error. The figures it produces are only as good as the equations.

What this does not establish. It does not show that the equations are wrong for any one patient. The bias is ICMR-NIN's statement about Indians as a group. It does not measure any patient's BMR. That needs a measurement under the conditions that define BMR, not a formula. And the 52 kcal/day step between age bands is a feature of the table, not of a body two years older.

12 · A column that is a weighted sum of other columns

EXERCISE 1. E is energy intake, in kilocalories a day. P, F and C are protein, fat and carbohydrate intake, in grams a day. A predictor is a column that a model uses to predict an outcome, here weight change, with one weight for each predictor. The 4, 9 and 4 are the Atwater general factors, in kilocalories a gram. 4P means 4 times P.

The first sentence says that on every row, E is a weighted sum of P, F and C, with weights 4, 9 and 4. So E is redundant beside them.

The second sentence asks for four separate weights, one for each predictor. The table cannot fix them. Take any set of weights. Move some weight off E and onto P, F and C in the proportions 4, 9 and 4. Every predicted value stays the same. “The weight on fat” has infinitely many values that fit equally well.

What they could have done: enter three of the four, and say which three. With E, P and C, the weight on E answers “energy, with protein and carbohydrate held fixed”, which can only move by moving fat. That is a real question, and it is a different question from the one asked with P, F and C.

1. Row by row, 3 times the entry of a plus 2 times the entry of b.

```
3 times 1 plus 2 times 4 = 11
3 times 2 plus 2 times 0 = 6
3 times 3 plus 2 times 1 = 11
```

The column is (11, 6, 11).

2. Each row gives one equation.

```
row 1: 1 times u plus 1 times v = 5
row 2: 2 times u plus 0 times v = 6
row 3: 3 times u plus 2 times v = 13
```

Row 2 holds u alone.

```
6 divided by 2 = 3
```

So u is 3. Put it into row 1.

```
5 minus 3 = 2
```

So v is 2. Row 3 is the check.

```
3 times 3 plus 2 times 2 = 13
```

It holds. So y is 3 times a plus 2 times b, a weighted sum with weights 3 and 2.

3. Check every row with the weights given.

```
2 times 2 plus 3 times 1 = 7
2 times 1 plus 3 times 3 = 11
2 times 4 plus 3 times 0 = 8
```

Rows 1 and 3 hold. Row 2 gives 11, and y has 8 there. One failing row is enough: y is not 2a + 3b.

4. Two weights, so use two rows to find them. Call them u and v.

```
row 1: 2 times u plus 1 times v = 4
row 2: 3 times u plus 2 times v = 7
```

Double row 1, so that v carries the same number in both.

```
2 times 2 = 4
2 times 4 = 8
```

That gives 4 times u plus 2 times v = 8. Subtract row 2 from it.

```
4 minus 3 = 1
8 minus 7 = 1
```

So u is 1. Put it into row 1.

```
4 minus 2 times 1 = 2
```

So v is 2. Now check the two rows you did not use.

```
row 3: 1 times 1 plus 2 times 5 = 11
row 4: 1 times 4 plus 2 times 3 = 10
```

Row 3 holds. Row 4 gives 10, and y has 9 there. So y is not a weighted sum of a and b. If you had stopped after row 3, you would have said it was.

5. Call the weight on a u, and the weight on the column of 1s w. Every entry of the column of 1s is 1, so w is simply added on each row.

```
row 1: 1 times u plus w = 8
row 2: 2 times u plus w = 11
```

Subtract row 1 from row 2. The w cancels.

```
2 minus 1 = 1
11 minus 8 = 3
```

So u is 3. Put it into row 1.

```
8 minus 3 = 5
```

So w is 5. Check row 3.

```
3 times 4 plus 5 = 17
```

It holds. So y is 3 times a, plus 5, on every row.

6. The Atwater general factors are 4 kcal a gram for protein, 9 for fat and 4 for carbohydrate (FAO 2003). Multiply each by its grams and add.

```
4 times 8 plus 9 times 12 plus 4 times 60 = 380
```

380 kcal, which is the label's figure. The energy on this label is exactly the Atwater weighted sum of its grams, so it tells you nothing the grams do not.

7. Call the weight on fat f and on carbohydrate c. Items A and B give two equations.

```
A: 10 times f plus 50 times c = 290
B: 20 times f plus 30 times c = 300
```

Double A, so that f carries 20 in both.

```
2 times 50 = 100
2 times 290 = 580
```

That gives 20 times f plus 100 times c = 580. Subtract B from it.

```
100 minus 30 = 70
580 minus 300 = 280
280 divided by 70 = 4
```

So c is 4. Put it into A.

```
290 minus 50 times 4 = 90
90 divided by 10 = 9
```

So f is 9. Check item C, which you did not use.

```
9 times 5 plus 4 times 70 = 325
```

It holds. The weights are 9 and 4 kilocalories a gram: kilocalories on the energy side, divided by grams on the other. They are the Atwater general factors for fat and carbohydrate (FAO 2003), which is what this made-up table was built from.

8. Call the weights u on EI and v on EE. Days 1 and 2 give two equations.

```
day 1: 2500 times u plus 2400 times v = 100
day 2: 2000 times u plus 2200 times v = -200
```

Make u carry the same number in both: multiply day 1 by 4 and day 2 by 5.

```
4 times 2500 = 10000
4 times 2400 = 9600
4 times 100 = 400
5 times 2000 = 10000
5 times 2200 = 11000
5 times -200 = -1000
```

Subtract the first new equation from the second. The u terms cancel.

```
11000 minus 9600 = 1400
-1000 minus 400 = -1400
-1400 divided by 1400 = -1
```

So v is minus 1. Put it into day 1.

```
100 plus 2400 = 2500
2500 divided by 2500 = 1
```

So u is 1. Check day 3.

```
1 times 2300 plus -1 times 2300 = 0
```

It holds. The weights are 1 and minus 1, both without a unit, since all three columns are in kilocalories a day. That is Hall's equation: dES/dt is EI minus EE, a weighted sum.

9. Energy is 4 times protein plus 9 times fat plus 4 times carbohydrate on every row (FAO 2003). So 0.01 times energy is the same as 0.04 times protein, plus 0.09 times fat, plus 0.04 times carbohydrate.

```
0.01 times 4 = 0.04
0.01 times 9 = 0.09
```

One set is (0.04, 0.09, 0.04, 0): all the weight on the grams. A second puts half on each: half of each gram weight, and half of 0.01 on energy.

```
0.04 divided by 2 = 0.02
0.09 divided by 2 = 0.045
0.01 divided by 2 = 0.005
```

The second set is (0.02, 0.045, 0.02, 0.005). Check all three on the row given.

```
0.01 times 460 = 4.6
0.04 times 10 plus 0.09 times 20 plus 0.04 times 60 = 4.6
0.02 times 10 plus 0.045 times 20 plus 0.02 times 60 plus 0.005 times 460 = 4.6
```

All three give 4.6. Any other split works too, so the table cannot choose between them.

10. The last line broke. The weights were found from rows 1 and 2, and those two rows were then taken as proof. Rows used to find the weights will always fit them; that is how they were found. The check is the rows that were left over, and nobody looked at them.

```
row 3: 1 times 3 plus 2 times 1 = 5
row 4: 1 times 5 plus 2 times 1 = 7
```

Row 3 holds. Row 4 gives 7, and y has 9 there. So y is not a weighted sum of a and b.

Why the wrong answer looks reasonable: the algebra is correct, and "the weights fit" sounds like a test that was passed. It was a test that could not fail.

11. The conclusion broke. Three weights were found from three rows, which left no row to check. The three rows will fit by construction, so the "check" repeats the rows used to solve.

Add a fourth made-up adult with BMR 8.0 MJ/day and PAL 2.0. The report's relation, "BMR times PAL is equal to TEE" (FAO/WHO/UNU 2004), gives that adult's TEE.

```
8.0 times 2.0 = 16
1.5 times 8.0 plus 6 times 2.0 minus 9 = 15
```

The weights give 15, and the adult's TEE is 16. They fail at the first row beyond the ones used to find them. TEE is BMR times PAL, a product, which is not a weighted sum of BMR, PAL and 1s.

Why the wrong answer looks reasonable: the solving is correct and every row "holds". With as many rows as weights, almost any column would have held.

12. The fourth line broke. The label prints energy as a whole number, so 367.2 is printed as 367. A gap of 0.2 kcal is the rounding, not a column with information of its own.

```
4 times 7.3 plus 9 times 11.6 plus 4 times 58.4 = 367.2
367.2 minus 367 = 0.2
```

The gap is below the half a kilocalorie that rounding to a whole number can move a figure by. The energy is the Atwater sum of the grams (FAO 2003), rounded. It is redundant beside them, and the last line would put the same fact into a model twice.

Why the wrong answer looks reasonable: the test was applied correctly, and a strict reading of it says one mismatch settles it. A mismatch smaller than the printing can show is not a mismatch.

13. What to compute: whether fat is a weighted sum of energy, protein and carbohydrate in this table. Energy is 4 times protein plus 9 times fat plus 4 times carbohydrate (FAO 2003). Take 4 times protein and 4 times carbohydrate away from energy, and divide by 9. So fat is the weighted sum of energy, protein and carbohydrate with weights 1/9, −4/9 and −4/9.

Check it on a made-up day: energy 2000 kcal, protein 60 g, carbohydrate 260 g.

```
2000 minus 4 times 60 minus 4 times 260 = 720
720 divided by 9 = 80
4 times 60 plus 9 times 80 plus 4 times 260 = 2000
```

Fat is 80 g, and the Atwater sum gives back the 2000 kcal. On every row, fat is fixed once the other three are known.

So "fat, with energy, protein and carbohydrate held fixed" describes rows the table cannot contain. Whatever number the model printed for fat is one of infinitely many that fit the table equally well.

What this does not establish. It does not show that fat is tied to weight gain, and it does not show that it is not. The study's sentence says nothing about fat either way. To ask about fat, drop energy and keep protein, fat and carbohydrate. Then the weight on fat answers "fat, with protein and carbohydrate held fixed", which lets energy move with fat. Say that out loud, because it is a different question.

14. What to compute: whether carbohydrate is a weighted sum of the columns the table holds. By the table's own rule it is 100 times the column of 1s, plus −1 times each of water, protein, fat and ash. Check it on one made-up row: water 12 g, protein 10 g, fat 2 g, ash 1 g.

```
100 minus 12 minus 10 minus 2 minus 1 = 75
```

75 g of carbohydrate. On every row where all four other columns are present, carbohydrate is fixed by them, with weights 100, −1, −1, −1 and −1. The colleague is right about the numbers: dropping it loses no value that cannot be written back in.

But only if every subtracted column is in the table. Take a copy without ash, or without water. Its carbohydrate column is then, in general, not a weighted sum of what is left. Run the test before you drop it.

What this does not establish. It does not show that the carbohydrate figure is right. Every error in the water, protein, fat and ash columns lands in it, because it is what is left over. It does not show what the carbohydrate is made of. FAO's report says Atwater's carbohydrate "is determined by difference, and thus includes fibre", so a by-difference figure is not sugar and starch alone. And "we lose nothing" is true only for a reader who knows the rule. Write the rule down beside the table if you drop the column.

13 · A random variable and its distribution

EXERCISE 1. Open with what they saw. The survey counted four households, and two had five members. Two out of four is a half. That is a fact about those four households.

Then the difference. The probability is a fact about the whole village: the share of all its households that have five members. The survey's half is an estimate of that share, and four households is a very small sample. Another four could easily give a quarter, or none.

Calibrate them. With only four households, the report cannot say whether the true share is nearer a quarter or a half. It can say five-member households were seen.

The sentence to quote: "Two of the four households surveyed had five members. A sample that small tells us little about how common that is in the village."

1. Check both rules. Each probability is between 0 and 1. Now add them.

```
0.1 plus 0.3 plus 0.4 plus 0.2 = 1.0
```

They add to 1, and the four values do not overlap. Yes, it is a probability distribution.

2. The four probabilities must add to 1. Add the three you have, and take the total from 1.

```
0.25 plus 0.35 plus 0.15 = 0.75
1 minus 0.75 = 0.25
```

P(X = 3) is 0.25.

3. Each coin lands heads (H) or tails (T), so there are four outcomes, each equally likely.

```
HH    HT    TH    TT
```

Each outcome has probability 1 in 4.

```
1 divided by 4 = 0.25
```

Now apply the rule "count the heads" to each outcome. HH gives 2. HT and TH both give 1. TT gives 0.

```
x P(X = x)
0 0.25
1 0.5
2 0.25
```

P(X = 1) is 0.5 because two outcomes give 1.

```
0.25 plus 0.25 = 0.5
0.25 plus 0.5 plus 0.25 = 1
```

The probabilities add to 1.

4. Add the probabilities of the values in each range.

```
0.2 plus 0.1 = 0.3
0.2 plus 0.4 plus 0.2 = 0.8
```

$P(X \geq 3)$ is 0.3, and $P(1 \leq X \leq 3)$ is 0.8.

The complement of "X is 0" is "X is 1 or more". Its probability is the sum over 1, 2, 3 and 4. Take that from 1.

```
0.2 plus 0.4 plus 0.2 plus 0.1 = 0.9
1 minus 0.9 = 0.1
```

$P(X = 0)$ is 0.1, which matches the table.

5. Add each row.

```
0.3 plus 0.3 plus 0.3 = 0.9
0.5 plus -0.1 plus 0.6 = 1.0
0.2 plus 0.5 plus 0.3 = 1.0
```

Table a adds to 0.9, not 1. It breaks the rule that the probabilities add to 1.

Table b adds to 1, but $P(X = 2)$ is minus 0.1. It breaks the rule that each probability is between 0 and 1. Adding to 1 does not rescue it.

Table c passes both rules. It is the only probability distribution of the three.

6. The area under the density must be 1. The rectangle is 4 wide, so its height is 1 divided by 4.

```
1 divided by 4 = 0.25
```

The probability of a range is the area over it: a rectangle 1.5 wide and 0.25 high.

```
2.5 minus 1 = 1.5
1.5 times 0.25 = 0.375
```

$P(1 \leq X \leq 2.5)$ is 0.375. $P(X = 3)$ is 0, because a single value has no width and so no area.

7. The density makes a triangle with base 2 and height 1. A triangle's area is half the base times the height, as in section 5.

```
0.5 times 2 times 1 = 1
```

The total area is 1.

At $x = 1$, halfway along, the straight line has risen halfway, to 0.5. So the area up to $x = 1$ is a smaller triangle with base 1 and height 0.5.

```
0.5 times 1 times 0.5 = 0.25
```

$P(X \leq 1)$ is 0.25. Half the range holds only a quarter of the probability, because the density is low on that side.

8. Each patient has a chance of 1 in 20. The probability of each value is the number of patients with it, divided by 20.

```
8 divided by 20 = 0.4
6 divided by 20 = 0.3
4 divided by 20 = 0.2
2 divided by 20 = 0.1
```

```
x (days) P(X = x)
2    0.4
3    0.3
4    0.2
7    0.1
```

Check the total.

```
0.4 plus 0.3 plus 0.2 plus 0.1 = 1.0
```

A stay of 4 days or more is X = 4 or X = 7.

```
0.2 plus 0.1 = 0.3
```

The probability is 0.3. The values of X are in days; the probabilities have no unit.

9. Rounding to the nearest 0.2 kg can be out by at most half of 0.2 either way.

```
0.2 divided by 2 = 0.1
```

X runs from −0.1 kg to 0.1 kg, a width of 0.2 kg. The height makes the area 1.

```
1 divided by 0.2 = 5
```

The height is 5 per kilogram.

“More than 0.05 away from zero” is two strips: from 0.05 to 0.1, and from −0.1 to −0.05. Each strip is 0.05 kg wide.

```
0.1 minus 0.05 = 0.05
0.05 times 5 = 0.25
0.25 plus 0.25 = 0.5
```

The probability is 0.5, under the flat model. Half the time, the rounding alone moves the reading by more than 50 grams.

10. Three of the five stayed 2 days.

```
3 divided by 5 = 0.6
```

The relative frequency is 0.6. For the whole ward, P(X = 2) is 8 divided by 20.

```
8 divided by 20 = 0.4
```

The ward’s 0.4 is the probability, fixed by the whole list of 20. The student’s 0.6 is a relative frequency from five patients, an estimate of 0.4. It misses by 0.2, and another five would give a different figure. None of the student’s five stayed 4 days, yet P(X = 4) is 0.2 in the ward.

11. The third line broke. The height of a probability density is not a probability. It is probability per kilogram, and the first sign of trouble is that 10 is bigger than 1, which no probability can be.

For a continuous variable, the probability of one exact value is the area over a strip of no width, which is 0.

```
0 times 10 = 0
```

P(X = 0.02) is 0. Ask about a range instead. For example, the chance of an error between 0.015 and 0.025 kg is a strip 0.01 kg wide.

```
0.025 minus 0.015 = 0.01
0.01 times 10 = 0.1
```

Why the wrong answer looks reasonable: the height is the only number the density gives at a point. So it is tempting to read it as "the chance at that point". It is how thickly the probability is packed there.

12. The fourth line broke. It treats a relative frequency from five patients as the probability for the ward. The ward's list says four patients stayed 4 days.

```
4 divided by 20 = 0.2
```

P(X = 4) is 0.2. Missing a value in a sample of five is not surprising. The chance that one random pick is not a 4-day patient is 0.8. If the five picks were independent, the chance of seeing no 4-day stay would be about a third.

```
0.8 times 0.8 times 0.8 times 0.8 times 0.8 = 0.32768
```

(The student picked five different patients, so the picks are not quite independent. The figure is a rough guide, not the exact chance.)

Why the wrong answer looks reasonable: "not seen" and "cannot happen" sound alike, and the sample was random. A random sample is fair. It is not complete.

13. The second line broke, and the table was broken before it. The bands "under 60 kg" and "50 to 70 kg" overlap: anyone from 50 to 60 kg is in both. The values of a probability distribution must not overlap.

You can see it in the total.

```
0.3 plus 0.5 plus 0.3 = 1.1
```

The three add to 1.1, not 1, because the people from 50 to 60 kg are counted twice. Adding the first two bands counts them twice again. The table cannot give P(under 70 kg) at all without the share from 50 to 60 kg, which it does not print.

Why the wrong answer looks reasonable: adding probabilities over a range is the right move for bands that do not overlap. It was applied to bands that do.

14. What to compute: the relative frequency the note is quoting.

```
12 divided by 40 = 0.3
```

The arithmetic is right: 0.3 of the patients surveyed weighed 80 to 90 kg.

What this does not establish. It is not the probability. It is an estimate of it, from 40 patients, and another 40 would give a different figure.

It is not even an estimate for "the next patient" unless the next patient comes from the same population, picked the same way. The survey may have been of one clinic, one season or one referral route. A sample says nothing about a population it was not drawn from, as Book 0, D6 warns.

And "between 80 and 90 kg" needs its ends settled. Does 90.0 on the scale count? With a scale that rounds, a displayed 90.0 covers true weights on both sides of 90.

What to write instead: "Of 40 patients surveyed, 12 (0.3) weighed 80 to 90 kg."

15. What to compute: the slide's figure only follows if PAL is spread evenly, with a flat density, from 1.40 to 2.40. Under that model, the density's height is 1 divided by the width.

```
2.40 minus 1.40 = 1.00
1 divided by 1.00 = 1
```

The height is 1 per unit of PAL. The area above 1.90 is a rectangle 0.50 wide.

```
2.40 minus 1.90 = 0.50
0.50 times 1 = 0.5
```

Under a flat model, the chance is 0.5. So the slide's arithmetic is right for the model it silently chose.

What this does not establish. The report gives a range, not a distribution. Nothing in the sentence says PAL values are spread evenly across it. The flat shape is the slide's assumption.

Worse, the sentence is about populations, not people. It says what "free-living adult populations" can sustain over a long time. A random adult's PAL is a different random variable from a population's PAL. Whether single adults sit inside or outside the range, the sentence does not say, and it gives no distribution for them at all.

And "about" means the ends themselves are rounded. What to say instead: "The report gives a range of about 1.40 to 2.40 for the PAL that adult populations can sustain. It gives no chance for any one adult."

14 · Expectation

EXERCISE 1. The first line in words: the expected value of X equals a sum. For every value x_i that X can take, multiply that value by the probability that X takes it. Then add up the products. The Greek capital sigma, Σ, means "the sum of", as in section 1. The i is a label that runs through the values, so x_i is the i-th value. P(X = x_i) is the probability that X equals x_i, and it has no unit. Each term x_i times P(X = x_i) is in kcal/day, so E(X) is in kcal/day.

In the first line, E is followed by a bracket holding a random variable, X. That E is an expected value. The paper writes it with round brackets; this book writes E[X].

In the second line, E carries the label I, for intake, and no bracket. EI is energy intake, and it is a rate, in kcal/day.

The test is what follows the E. A label means energy. A bracket around a random variable means an expected value.

1. Multiply each value by its probability, and add.

```
1 times 0.2 plus 2 times 0.5 plus 3 times 0.3 = 2.1
```

E[X] is 2.1.

2.

```
-2 times 0.5 = -1
1 times 0.25 = 0.25
4 times 0.25 = 1
-1 plus 0.25 plus 1 = 0.25
```

E[X] is 0.25. X takes only −2, 1 and 4, so the expected value is not a value X can take.

3. Use E[aX + b] = aE[X] + b each time.

```
2 times 3 plus 5 = 11
-1 times 3 = -3
3 minus 3 = 0
```

E[2X + 5] is 11. E[−X] is minus 3, with a = −1 and b = 0. E[X − 3] is 0, with a = 1 and b = −3.

4. The sum rule needs no independence, so nothing more is needed. Use E[aX + bY] = aE[X] + bE[Y].

```
4 plus -1 = 3
4 minus -1 = 5
3 times 4 minus 2 times -1 = 14
```

E[X + Y] is 3, E[X − Y] is 5 and E[3X − 2Y] is 14.

5. Call P(X = 10) q. Then P(X = 0) is 1 minus q, and E[X] is 0 times (1 minus q) plus 10 times q. The first term is 0, so 10 times q is 2.5.

```
2.5 divided by 10 = 0.25
```

P(X = 10) is 0.25. Check it.

```
0 times 0.75 plus 10 times 0.25 = 2.5
```

6. With PAL fixed, TEE is 1.75 times BMR, so this is E[aX] with a = 1.75.

```
1.75 times 7.10 = 12.425
```

E[TEE] is 12.425 MJ/day. The report prints 12.42, which is the same figure cut to two decimal places.

The rule works here only because PAL is a fixed number. If PAL were a random variable that moved with BMR, E[PAL times BMR] would not in general be E[PAL] times E[BMR].

7. The expected value of a sum is the sum of the expected values, with or without independence. Every day has the same expected value, so multiply.

```
7 times -150 = -1050
28 times -150 = -4200
```

Minus 1,050 kcal over 7 days and minus 4,200 kcal over 28 days. The link between days changes how widely the totals spread. It does not change these expected values.

8. The logged surplus is the true surplus plus 120. Use E[X + b] = E[X] + b.

```
30 plus 120 = 150
7 times 150 = 1050
7 times 30 = 210
1050 minus 210 = 840
```

The log expects 150 kcal a day and 1,050 kcal a week, against a true 210 kcal a week. The gap is 840 kcal a week, which is 7 times the 120. More weeks of logging repeat it every week.

9.

```
4 plus 7 plus 5 plus 5 plus 9 = 30
30 divided by 5 = 6.0
```

The survey's mean is 6.0 people.

E[X], 5.0, is the parameter. It is fixed by the whole village and the way you pick. The 6.0 is a statistic from five households, and it estimates 5.0. It misses by 1.0, and the first survey's 5.75 missed by less. Neither survey changes the village.

With a much bigger random sample, picking with replacement (a household can be picked more than once), the sample mean would tend to land closer to 5.0. That is the law of large numbers. It says what tends to happen over many picks. It does not say that any one survey will land close.

10. The second line broke. It averages the four values as if each had probability 0.25. They do not. Weight each by its own probability.

```
-300 times 0.2 plus 0 times 0.3 plus 200 times 0.4 plus 500 times 0.1 = 70
```

E[X] is 70 kcal a day, not 100.

Why the wrong answer looks reasonable: it is the ordinary mean of Book 0, D5, a move the reader has done a hundred times. It is right when every value is equally likely. The probabilities were given, and the working never used them.

11. The second line broke. The week's surplus is the sum of the seven days, not their average. So its expected value is the sum of the seven expected values.

```
70 plus 70 plus 70 plus 70 plus 70 plus 70 plus 70 = 490
```

The expected surplus for the week is 490 kcal. The working found the expected surplus per day over the week, 70 kcal a day, and then labelled it as a weekly total.

Why the wrong answer looks reasonable: "the expected value is an average" is true, and it invites averaging again. The units give it away. 70 kcal a day and 490 kcal over a week are different quantities.

12. The last line broke. BMI is a ratio, and E[weight divided by height squared] is not E[weight] divided by E[height] squared. The rules for expected values cover sums and fixed multiples. They do not cover a ratio or a square. Go back to the distribution: work out each adult's BMI, then weight by probability.

```
1.5 times 1.5 = 2.25
50 divided by 2.25 = 22.22
1.8 times 1.8 = 3.24
90 divided by 3.24 = 27.78
0.5 times 22.22 plus 0.5 times 27.78 = 25.0
```

E[BMI] is 25.0 kg/m^2, not 25.7.

Why the wrong answer looks reasonable: every line before the last is correct, and "put the averages into the formula" works for sums. It fails as soon as the formula divides or squares.

13. What to compute: the expected value of the model, and then the chance of the thing the leaflet promises.

```
0 times 0.3 plus 4 times 0.4 plus 8 times 0.3 = 4.0
```

The expected loss is 4.0 kg, so the first sentence matches the model. The chance of losing exactly 4 kg in this model is 0.4. The chance of losing nothing is 0.3.

What this does not establish. “You will lose 4 kg” is a claim about one person. The expected value is an average over many, and here three in ten lose nothing. A person who loses nothing has not failed the programme’s arithmetic; they are one of its outcomes.

It also does not say where the model came from. Perhaps the 0.3, 0.4 and 0.3 came from one group of participants. Then they are relative frequencies, and 4.0 is a sample mean that estimates the true expected loss. And it says nothing about later on: an average at one time point says nothing about weight in the months after it.

What to say instead: “On average people lost 4 kg. Three in ten lost none, and three in ten lost 8 kg.”

14. What to compute: the talk treats the extra 100 kcal of intake as a surplus of 100 on every day. Then the expected surplus over 365 days is 365 times 100.

```
365 times 100 = 36500
```

36,500 kcal. The arithmetic of the expected value is right on that condition.

What this does not establish. The condition is the whole claim. The sum rule gives 365 times 100 only if every one of the 365 days has an expected surplus of 100. Book 1 shows why that fails. As stores and weight rise, expenditure rises with them (Book 1, section 4). So the same intake gives a shrinking surplus. This is the static rule of Book 1, section 7, and it shares that rule’s flaw.

It is also an expected value, not a forecast for one person. Real days scatter around any expected value, and so do real years.

And “all of it is stored” is the energy-balance identity of Book 1, section 5 applied to the total. That part is right by definition, whatever the surplus turns out to be. What is wrong is the 36,500.

15 · Variance, and why independent errors add as squares

EXERCISE 1. X is the error in measured intake and Y the error in measured expenditure, each in kcal a day. X − Y is the error in the measured imbalance, intake minus expenditure, also in kcal a day.

Var( ) is the variance: the expected squared deviation from the expected value. So each of the three terms is in square kcal a day. That is kcal a day multiplied by itself. So no term can be set beside a daily intake until you take its square root.

The line needs X and Y to be independent: knowing one error tells you nothing about the other. If they are not, a correction term, the covariance, is missing.

The minus sign becomes a plus because X − Y is 1 times X plus minus 1 times Y. The rule squares each constant, and minus 1 times minus 1 is 1. A spread is never reduced by subtracting.

EXERCISE 2. Energy balance is intake minus expenditure, and each is measured with error. The best method for expenditure, doubly labelled water, is uncertain by more than 100 kcal a day, Hall and colleagues report. Intake from what people report is worse.

When you subtract one uncertain number from another, the uncertainties do not cancel. They combine, and the result is rougher than either. So a 50 kcal change in one person sits well inside the noise.

Quotable sentence: “Take one rough measurement away from another and the answer gets rougher, not sharper.” A 50-calorie change in one person is lost in that roughness.

Calibration: the arithmetic is settled. What a study can do instead, averaging over many people, is a separate question, and that is where the design has to earn its answer.

1. Expected value: each value times its probability, added up.

```
2 times 0.25 plus 4 times 0.5 plus 6 times 0.25 = 4
```

Deviations from 4 are minus 2, 0 and 2. Their squares are 4, 0 and 4. Weight them and add.

```
4 times 0.25 plus 0 times 0.5 plus 4 times 0.25 = 2
```

The variance is 2. The standard deviation is the square root of 2, about 1.414.

2. Expected value:

```
0 times 0.8 plus 10 times 0.2 = 2
```

Deviations from 2 are minus 2 and 8. Their squares are 4 and 64.

```
4 times 0.8 plus 64 times 0.2 = 16
```

The variance is 16. The standard deviation is the square root of 16, which is 4.

3. The standard deviation is the square root of the variance, so the variance is the standard deviation times itself.

1. The square root of 9 is 3.
2. The variance is 0.5 times itself.

```
0.5 times 0.5 = 0.25
```

1. The square root of 0.49 is 0.7, because:

```
0.7 times 0.7 = 0.49
```

4. Use $\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$. The added number moves nothing in the spread.

For $3X + 7$, a is 3.

```
3 times 3 times 4 = 36
```

The variance is 36 and the standard deviation is 6. That is 3 times the standard deviation of X, which is 2.

For $-2X + 5$, a is minus 2, and minus 2 times minus 2 is 4.

```
4 times 4 = 16
```

The variance is 16 and the standard deviation is 4, which is 2 times 2. The minus sign turns the values round but does not shrink their spread.

5. The variances add, for a sum and for a difference alike.

```
9 plus 16 = 25
```

$\mathrm{Var}(X + Y)$ and $\mathrm{Var}(X - Y)$ are both 25. Both standard deviations are the square root of 25, which is 5.

The standard deviations of X and Y are 3 and 4.

```
3 plus 4 = 7
```

Seven is not 5. Adding standard deviations overstates the spread of the sum.

6. Each variance is 2 times 2, which is 4. The variances of independent parts add.

```
9 times 4 = 36
```

The sum's standard deviation is the square root of 36, which is 6. That is 2 times the square root of 9.

The mean is the sum times one ninth. Its standard deviation is one ninth of the sum's: 6 divided by 9, about 0.667. That is 2 divided by the square root of 9, the square-root law.

7. The total's standard deviation is 10 times the square root of the number of days. Set it equal to 30 and rearrange.

```
30 divided by 10 = 3
```

The square root of the number of days is 3, so the number of days is 3 times 3, which is 9.

Check it forwards: 9 days of variance 100 give a variance of 900, and the square root of 900 is 30.

8. 1. 5% is 0.05. Divide the uncertainty by 0.05.

```
100 divided by 0.05 = 2000
```

An expenditure of 2,000 kcal a day. At any higher expenditure, 5% is more than 100 kcal a day, which fits the paper's "more than 100".

1. Multiplying every value by 4.184 multiplies the standard deviation by 4.184.

```
100 times 4.184 = 418.4
```

418.4 kJ a day, which the paper rounds to 420.

1. The variance is multiplied by 4.184 squared.

```
4.184 times 4.184 = 17.505856
```

About 17.5. The variance in square kJ a day is about 17.5 times the variance in square kcal a day.

9. The imbalance is intake minus expenditure. Independent variances add.

```
100 times 100 plus 250 times 250 = 72500
```

The standard deviation is the square root of 72,500, about 269.3 kcal a day.

It is a floor. The 100 was a floor, and a larger expenditure error would make the total larger. Label it "at least 269 kcal a day".

10. One day's variance is 72,500, from the last problem. Seven independent days:

```
7 times 72500 = 507500
```

The square root of 507,500 is about 712.4 kcal over the week. That is 269.3 times the square root of 7, which is about 2.646.

The average is the total divided by 7.

```
712.4 divided by 7 = 101.8
```

About 101.8 kcal a day, which is 269.3 divided by the square root of 7. Both are floors, because 269.3 was.

Both figures rest on the days being independent. A reporting habit that repeats every day breaks that, and its part of the error would not shrink.

11. The variances add, so the intake variance is the total variance minus the expenditure variance.

```
1000 times 1000 minus 100 times 100 = 990000
```

The square root of 990,000 is about 995 kcal a day. The paper's expenditure error is more than 100, so 995 is the most the intake part could be.

So almost all of the combined error would come from intake. Take the expenditure error away entirely and the total would fall only from 1,000 to about 995.

Two things here are readings, not facts from the paper. The paper does not say that its 1,000 is a standard deviation, or how it combined the two errors.

12. The third step broke, twice over. Standard deviations are not combined by subtracting, and they are not combined by adding either. Variances add, for a difference as for a sum.

```
300 times 300 plus 100 times 100 = 100000
```

The square root of 100,000 is about 316.2 kcal a day, not 200.

Why the wrong answer looks reasonable: the working follows the shape of the quantity. The imbalance is a subtraction, so subtracting the errors feels faithful to it. But a minus sign in the quantity does not reach its spread. The rule squares the minus 1 away.

13. The second line broke. Multiplying X by 3 multiplies its variance by 3 times 3, not by 3.

```
3 times 3 times 4 = 36
```

Var(3X) is 36 and its standard deviation is 6, three times X's standard deviation of 2.

Why the wrong answer looks reasonable: multiplying by 3 does multiply the expected value by 3, and the error borrows that rule. A quick check catches it. Tripling every value must triple every distance from the middle, so the standard deviation, a distance, triples.

14. The second line broke. The square-root growth holds for independent errors, and this error is the same every day. It is systematic, not random. It adds in full each day.

```
300 times 30 = 9000
```

The error in the 30-day total is 9,000 kcal, not about 1,643. In the average day it is still 300 kcal, however many days you record.

Why the wrong answer looks reasonable: the square-root law has just been taught. It is the right tool for how a total's spread grows. The question the line skipped is whether the days are independent. A repeated under-count makes every day's error the same, which is as far from independent as errors can be.

15. The third line broke. Subtracting the empty bowl's reading removes the bowl's true weight, but it does not remove that reading's random error. The two readings' random errors are independent, so their variances add.

```
7 times 7 plus 7 times 7 = 98
```

The portion's standard deviation is the square root of 98, about 9.9 grams, not 7.

Why the wrong answer looks reasonable: weighing by difference really does remove one kind of error. It removes a bias the two readings share, such as a scale that always reads 5 grams heavy. The working took that true fact and applied it to the random part, which is not shared.

16. What to compute: what the adding rule would give, and whether the rule applies here.

```
11.3 times 11.3 plus 16.0 times 16.0 = 383.69
```

The square root of 383.69 is about 19.6 pounds. The reader's arithmetic is right. The means also agree, to rounding.

```
27.6 minus 20.1 = 7.5
```

The paper prints 7.4. The gap of 0.1 is rounding: each mean was rounded before you subtracted them.

The rule does not apply, because the two figures are not independent. Each person's predicted loss and actual loss come from the same person and the same deficit, so they tend to rise and fall together. When two quantities rise and fall together, their difference spreads less than the adding rule says. A reported 12.6 against the rule's 19.6 is what such paired figures can look like.

What this does not establish. It does not show that 12.6 is correct. It shows only that 19.6 is not the figure to check it against. It does not say how strongly the two were related; that needs the covariance, which a later book, on probability and statistical inference, teaches. And it says nothing about whether the 3500 kcal rule is right, which is a different question.

17. What to compute: how the error of a total of 12 items grows under the two kinds of error.

Read "within 50 kcal" as a standard deviation of 50 kcal per item. If the items' errors are independent, the variances add.

```
12 times 50 times 50 = 30000
```

The square root of 30,000 is about 173.2 kcal for the day. That is 50 times the square root of 12, more than three times the advert's figure.

If every entry is off the same way, say each under-counts by 50 kcal, the errors add in full.

```
12 times 50 = 600
```

That would be 600 kcal for the day.

So the day's log is not accurate to within 50 kcal under either reading. The per-item figure does not carry over to the total.

What this does not establish. The advert does not say what "within 50 kcal" means: a standard deviation, a largest error, or something else. The 173 and the 600 are the two ends of a range, not a measurement. And none of it touches a larger gap. Was the portion entered the portion eaten? Is the energy on the label what this person absorbs?

18. What to compute: first whether the deficit column is baseline minus target at all, and only then the spread.

Check the means. If every person's deficit were their baseline minus their target, the mean deficit would be the mean baseline minus the mean target.

```
2876 minus 1409 = 1467
```

The paper reports 1,439, not 1,467. So the deficit column was not made by that subtraction for every person. It may come from a different definition, or from a different set of people, and the sentence does not say which.

The reviewer's spread, taken on its own terms:

```
484 times 484 plus 569 times 569 = 558017
```

The square root of 558,017 is about 747 kcal a day.

What this does not establish. The 747 is not a figure the 784 must match. The means already show the subtraction was not the one used. Even if it were, the rule needs baseline and target to be independent. If each target was set from the person's own baseline, the two are not independent. That kind of link would pull the spread below 747, not push it above. So dependence of that kind cannot explain the 784 either; the means have already shown the column was made some other way. What the check does show is how to read such a table: test the means before you test the spreads.

16 · Following a derivation without skipping

EXERCISE 1. Write the seven lines of the rebuild, each with its move beside it.

```
dES/dt = EI - EE (the identity)
dES/dt = 3500 × dW/dt (assumed: 3,500 kcal in every pound)
3500 × dW/dt = EI - EE (substitute)
3500 × dW/dt = ΔEB (assumed: the gap never changes)
dW/dt = ΔEB / 3500 (rearrange)
W(t) - W0 = ΔEB × t / 3500 (integrate from 0 to t)
W(t) = W0 + ΔEB × t / 3500 (rearrange)
```

Then box the two assumed lines in a different colour. Say that the first line is physics and the other four are safe moves, so the rule is exactly as good as those two lines.

Take the second box. Ask the resident what happens to expenditure as weight falls. It falls, because a smaller body spends less. So the gap in that line shrinks, and weight loss slows. That is the panel's reason, in its own words: weight change "will slow over time due to passive compensatory changes in energy expenditure".

Take the first box last. It is also an assumption: what a pound of lost weight is made of. Book 1 showed that a pound of fat and a pound of lean tissue hold very different energy.

End on the sign. Thomas and colleagues print a minus where the rebuild has a plus. The resident should say why before you do: their ΔEB must be the deficit, expenditure minus intake.

EXERCISE 2. The first law gives only one line of the derivation, the balance. The rest are premises.

Expenditure is drawn as a straight line in weight; the paper calls Equation 1 "a linearization". Equation 1 has no term for a gap between intake and expenditure at the start, so it takes weight as steady then. And the change in physical activity is set to zero where it was not measured. The paper says this makes the calculated intake changes possibly "in error, especially at the individual level".

For a group the paper puts the bias from that last premise at "<40 kcal/day when studying >100 individuals". That is a bound for a large group's mean, not for any one person. It comes from an earlier validation study this slide has not examined.

So the equation is exactly as good as its premises. It is not exact, and the paper does not claim it is.

1. Line 2 subtracts 6 from both sides.

```
26 minus 6 = 20
```

Line 3 divides both sides by 4.

```
20 divided by 4 = 5
```

Check by putting 5 back into line 1.

```
4 times 5 plus 6 = 26
```

2. Substitute: line 2 says x equals 5, so put 5 in place of x in line 1.

```
3 times 5 = 15
```

3. Three moves are missing.

```
3x - 4 = 11 (subtract 2x from both sides)
3x = 15 (add 4 to both sides)
x = 5 (divide both sides by 3)
```

Check in the first line: both sides give 21.

```
5 times 5 minus 4 = 21
2 times 5 plus 11 = 21
```

4. Differentiate both sides with respect to t. The rule is that a straight line's derivative is its slope, from section 3: the 4t gives 4 and the constant 3 gives 0. The rate of change of y is 4 at every t.

5. Integrate both sides from 0 to 10. The rate is constant, so the integral is rate times time.

```
W(10) - W(0) = -0.2 times 10 (integrate both sides from 0 to 10)
W(10) - 80 = -2 (substitute W(0) = 80)
W(10) = 78 (add 80 to both sides)
```

```
-0.2 times 10 = -2
-2 plus 80 = 78
```

6. Line 2 does not follow. Dividing both sides by 4 divides the whole left side, both of its terms, the rule from Book 0, C2. The 8 was left undivided.

```
x + 2 = 5 (divide both sides by 4)
x = 3 (subtract 2 from both sides)
```

Check in line 1.

```
4 times 3 plus 8 = 20
```

Line 3 of the original is a correct move from line 2. The whole error came in at one line, which is why you check every line and not only the last.

7. ΔEB is read as expenditure minus intake, so it is 700.

```
700 times 28 divided by 3500 = 5.6
200 minus 5.6 = 194.4
```

W(28) is 194.4 pounds on this rule. Under the other reading, ΔEB would be minus 700. The printed equation would then give 205.6 pounds, a rise during a deficit. That tells you the reading is wrong.

```
200 plus 5.6 = 205.6
```

8. Divide both sides by Δt. A side that is a difference is divided as a whole, so each term is divided.

```
ΔES / Δt = (energy taken in over the interval) / Δt - (energy spent over the inter-
val) / Δt (divide both sides by Δt)
```

Each side is now in kcal a day. The left is the average rate of change of stores over the interval. The right is the average rate of intake minus the average rate of expenditure.

Now shrink Δt towards zero. Each average rate settles on the rate at an instant, as in section 2.

```
dES/dt = EI - EE (shrink the interval: take the limit)
```

Every term is in kcal a day. This is the form Hall and colleagues write as ES = EI − EO, where their ES stands for the rate, dES/dt, and their EO is expenditure, EE.

9. Same premises: expenditure a straight line near the start, intake written as start plus change, weight steady at the start with U zero then.

```
EE(W) = EE0 + ε × (W - W0) (assumed: a straight line near the start)
EI = EI0 + ΔEI (definition)
ρ × dW/dt = EI0 + ΔEI - EE0 - ε × (W - W0) - U (substitute)
ρ × dW/dt = ΔEI - ε × (W - W0) - U (assumed: steady at the start, so EI0 - EE0 = 0)
ΔEI = ρ × dW/dt + ε × (W - W0) + U (add ε × (W - W0) and U to both sides, swap
sides)
```

With Δδ = 0 the paper's Equation 1 is ΔEIi = ρ × dBWi/dt + ε × (BWi − BW0) + UGE. The label i numbers the measurement interval. It is the same form.

One check the derivation forces: every term must be in kcal a day. The paper assumes the glucose lost is about 90 grams a day, a mass. Before it goes into Equation 1 it has to be turned into energy a day.

10. Line 2 broke. Dividing both sides by 3,500 divides the whole right side, not only its last term, the rule of Book 0, C2.

```
dW/dt = (EI - EE) / 3500
2000 minus 2500 = -500
```

Minus 500 divided by 3,500 is about minus 0.143 pounds a day.

Why the wrong answer looks reasonable: line 2 looks like line 1 with the 3,500 moved across. And the arithmetic in line 3 is done correctly. A unit check catches it at once. EI is in kcal a day, and EE divided by 3,500 is in pounds a day. Two terms in different units cannot be subtracted, so line 2 cannot have followed.

11. The last line broke, or rather it needed a premise nobody wrote. Rearranging line 3 gives:

```
ΔEI = ρ × dW/dt + ε × (W - W0) - (EI0 - EE0)
```

Dropping the last term needs EI0 equal to EE0: weight steady at the start. If weight was already falling or rising at the start, that term is not zero, and the equation gives the wrong change in intake.

Why the wrong answer looks reasonable: the result is the equation Polidori and colleagues use, so it looks confirmed. It is confirmed only under the premise. Write “assumed: steady at the start” beside the line, and a reader can check it against the baseline weights.

12. Line 2 broke. “Rate times time” is the rectangle, which works only for a rate that stays the same. This rate changes along a straight line, so the area under it is a trapezium. As section 5 taught, that is the mean of the starting and ending rates, times the time.

```
-0.1 plus -0.04 = -0.14
-0.14 divided by 2 = -0.07
-0.07 times 60 = -4.2
```

The change is minus 4.2 kg, not minus 2.4.

Why the wrong answer looks reasonable: every line of arithmetic is right, and the rule “rate times time” was correct in the section’s own derivation. There the rate was constant by assumption. Here it is not, and the move stopped applying without anyone saying so.

13. What to compute: the colleague’s figure, and then whether it is the figure the paper would get.

```
1439 times 64.8 = 93247.2
93247.2 divided by 3500 = 26.64
```

The colleague’s arithmetic is right: about 26.6 pounds.

But follow the derivation. The rule gives each person a prediction: their own deficit times their own days, divided by 3,500. The paper’s 27.6 is the mean of those predictions. The colleague took the mean deficit times the mean duration. The mean of products is not in general the product of means. They match only when deficit and duration do not vary together. If the people with larger deficits were also treated for longer, the mean of the products is larger, which would fit the gap here.

So the two figures are different quantities, and the gap is not evidence of a slip.

What this does not establish. It does not show that the paper’s 27.6 is right; you would need each person’s figures. It does not show how deficit and duration were related. Nor does it show that the predictions used the prescribed deficit; if they used each person’s achieved intake, the gap has a second source. And it says nothing for or against the 3500 kcal rule itself. The paper tests that by setting these predictions beside the losses people actually had.

14. What to compute: what the rebuilt rule gives, and which of its lines are physics.

```
500 times 7 = 3500
3500 divided by 3500 = 1
```

On the rule, 3,500 kcal a week is one pound a week. So the column’s arithmetic matches the static rule.

Now read the derivation. Only line 1, the identity, is physics. “A pound” needs line 2, that every 3,500 kcal is a pound of weight. “For as long as you keep it up” needs line 4, that the gap between intake and expenditure never changes. Hall and colleagues state that weight change “will slow over time” because expenditure falls as weight falls. So line 4 fails, and with it the steady pound a week.

What this does not establish. It does not tell you how much a real person loses in a week, a month or a year. That needs a model in which expenditure changes with weight. It also does not say whether they can hold the 500 kcal cut, which is a separate question about appetite.

17 · Four kinds of equation, and the notebook entry

EXERCISE 1. The kinds, and the one thing each entry must not leave out.

1. Identity. Counted across one boundary, the body, for one interval. Hall and colleagues call it "a statement of the principle of energy conservation". Their ES here is the rate, which this book writes dES/dt. EO is this book's EE. Names no cause.
2. Definition. PAL has no existence apart from this ratio. Units cancel: PAL has none.
3. The same definition, rearranged, used to estimate. Its kind is definition, but its answer is only as good as the BMR (fitted) and the PAL (measured or adopted) put in. ICMR-NIN's 2020 note changed both inputs for Indian adults.
4. Fitted. Schofield's data, 829 women in this band, by the table's count. Can run 10 to 12% high for Indians, per ICMR-NIN. Check the units: 14.818 is kcal a day per kg.
5. Fitted and then adopted. The factors are averaged from measurements, rounded, and used "regardless of the food", FAO says, as Book 0, E2 taught. Indian labelling rules then fix the factors by regulation, as Book 0, B4 showed.
6. Model assumption. As section 16 showed, it is the identity plus two assumptions. One is 3,500 kcal in every pound. The other is a gap that never changes. ΔEB must be read as expenditure minus intake for the printed minus sign to give a loss. W is in pounds.
7. Model. The identity plus premises, as section 16 rebuilt it. The paper calls it "a linearization". β "accounts for the adaptation of energy expenditure during a diet perturbation". δ is its physical activity parameter: "The baseline physical activity parameter was δ0 and Δδ represents changes in physical activity".
8. Model assumption with a fitted constant, kP, about 100 kcal a day per kg, fitted to group means. The authors say they do not yet know if it holds across a range of losses.
9. Model assumption. The authors "also investigated" it as an alternative. Its kI of 1 kcal a day per kg per day was set for the simulation. The paper's own words give no fit for it.
10. Model assumption (a model's output, rounded). The panel calls it "a best-case scenario" "for the average overweight person". It inherits every assumption of that model. It describes the eventual weight, not the time to reach it.

A full entry, for number 8, is in this section's illustration. The others follow its pattern.

EXERCISE 2. The answer first: the requirement is a calculation, and ICMR-NIN changed two of its inputs. Nobody's body had to change.

The requirement is the resting energy a body spends, its basal metabolic rate, times a number for how active people are. That number is the physical activity level. ICMR-NIN's 2020 note states the formula: TEE = BMR × PAL.

The first input. The resting figure comes from equations built on measurements of other populations. ICMR-NIN found they can run 10 to 12% high for Indians. It had already cut them by 5% and cut them by another 5% in 2020.

The second input. For sedentary adults the activity number fell from 1.53 to 1.40. The note's sentence calls it PAR, the physical activity ratio, but its heading and figures are the physical activity level. ICMR-NIN says the energy spent on each activity is lower for Indians. On its own, at the same resting figure, that lowers the requirement by about 8.5%.

```
1.40 divided by 1.53 = 0.915
```

What this means for policy. Any scheme norm still built on the 2010 figures uses a higher estimate than ICMR-NIN now gives. Check such norms against the 2020 figures. Next time the inputs are revised, the requirement will move again. Tie every requirement quoted in a scheme to the year of the inputs behind it.

Glossary

Every term of art this book teaches, in the plain words it gets where it is first used, and the section that teaches it.

Term	Plain words	Section
antiderivative	a function whose derivative is the given function; checked by differentiating it back, and written with plus C on the end, where C is any constant	6
average value (of a rate)	the integral divided by the length of the interval	5
baseline	the start: the weight at the start is the weight at baseline, written BW0	1
coefficients	the fixed numbers that a formula multiplies by	11
constant multiple rule	the derivative of k times a function is k times the derivative of the function	3
constant rule	the derivative of a constant is zero	3
continuous (random variable)	can take any value in an interval, and its probabilities are areas under a curve; the probability of any one exact value is 0	13
counter (of a sum)	the letter, usually i, that comes with Σ, with its first and last values; it only says where a term sits	1
critical point	an input inside the domain, not at one of its ends, at which the derivative is zero or does not exist	9
d/dx	said "dee by dee x": the derivative with respect to x of what follows	3
data matrix	a matrix in which each row is one case and each column is one kind of measurement, always in the same place	11
definition (kind of equation)	an equation that brings in a new quantity as a combination of others; it cannot be false	17
derivation	a chain of equations in which each line follows from lines above it by one move that can be named	16
differential equation	an equation involving an unknown function and one or more of its derivatives: a rule that gives the rate from the state	7
discrete (random variable)	the values it can take can be listed, one by one	13
dot product	multiply the entries of two vectors of the same length in matching positions, then add the products; written a · b, said "a dot b"	10

Term	Plain words	Section
e (the number)	the base of the one exponential curve whose slope where it crosses the vertical axis is exactly 1; about 2.71828	4
energy density	energy per kilogram	1
entry (of a vector or matrix)	one number in the list; in a matrix, found by its row first and its column second	10
equilibrium (steady state)	a constant value of the state at which the rule gives a rate of zero, written W∗ and read "W star"	8
expected value (expectation), E[X]	each value times its probability, all added up; the long-run average	14
first derivative test	sorts a critical point by the sign of the derivative either side: positive to negative is a local maximum, negative to positive a local minimum, the same sign neither	9
first-order (differential equation)	gives the rate of change of a quantity at a moment as a rule in its state, in the time, or in both	7
fitted equation	an equation whose form was chosen by its author and whose constants were estimated from one set of measurements	17
forward step (Euler's method)	from the state at time t, compute the rate from the equation, then take the state at t + h to be about the state at t plus h times that rate	7
fundamental theorem of calculus (evaluation theorem)	if f is continuous and F is any antiderivative of f, the integral of f from a to b is F(b) minus F(a)	6
half-time	the time a shrinking exponential takes to fall to half of any starting value: ln 2 divided by k	4
identity (kind of equation)	an equation that holds for every value of its symbols because a conservation law, or counting, requires it	17
initial value	the state at one stated time, which picks out one solution of a differential equation	7
initial-value problem	a differential equation and an initial value together; a function solves it only if it satisfies both	7
integrand	the rate being added up in an integral, written after the ∫	5
law of large numbers	repeat the chance process many times, and the average of the values the random variable takes settles towards its expected value	14
least squares	choosing a number by making squared misses as small as possible	9
length (of a vector)	the number of entries it has	10

Term	Plain words	Section
limit	the number the average rates close in on as the step h is taken closer and closer to zero, without h ever being set to zero	2
limits of integration	the two times at the foot and the top of the ∫ sign: where the time starts and where it stops	5
local maximum, local minimum	a top that is higher than the ground near it; a bottom that is lower than the ground near it	9
matrix	a rectangular table of numbers, named by a capital letter: a table with the labels taken off	11
matrix product	A times B, written AB: the entry in row i and column j is row i of A dotted with column j of B; defined only when A has as many columns as B has rows	11
model assumption	a claim, adopted by the author, about how a body or a population behaves; it could be false	17
move (in a derivation)	one named step from a line to the next: substitute, rearrange, divide by a time interval and shrink it, differentiate or integrate both sides, or assume	16
natural logarithm (ln)	the power to which e must be raised to give the number; defined only for numbers above zero	4
net change theorem	the integral of a rate over an interval is the change in the stock over that interval	5
net signed area	the area under a rate line with the area below the time axis counted as negative, so subtracted	5
notebook entry	one equation recorded in six parts: the equation as printed, each symbol in words with its unit, the whole equation as one sentence, its kind, what it assumes or whom it was fitted on, and the paper with the equation's number or page	17
overall maximum, overall minimum	the highest, or lowest, point over the whole stretch of inputs allowed	9
power rule	for a positive whole number n, the derivative of x^n is n times x^(n minus 1)	3
predictor	one of the several columns a study predicts an outcome from, each given one weight	12
probability density	a curve that never goes below zero and has a total area of 1 beneath it; the area over a range is the probability of that range, and its height is a probability per unit, not a probability	13
probability distribution	the list of every value a random variable can take, each paired with its probability; a model of the chance process, where a frequency table is data	13
quadratic	a function of the form $px^2 + qx + r$	9

Term	Plain words	Section
random variable	a rule that gives a number to each outcome of a chance process; written as a capital letter, such as X	13
rate constant (k)	the rate divided by the amount; its unit is the rate's unit divided by the amount's, such as per day	4
redundant (column)	a column that is a weighted sum of other columns: once they are known, it says nothing about any row that they did not already say	12
scalar	a single number used to multiply every entry of a vector	10
second derivative	the derivative of the derivative, the rate of change of the rate; written f″(x)	3
second derivative test	where the derivative is zero, a second derivative above zero means a local minimum, below zero a local maximum, and zero decides nothing	9
set point model	Hall and Guo's name for a model with feedback on both intake and expenditure; its equilibrium still moves when intake is shifted	8
settling point model	Hall and Guo's model in which intake is flat and expenditure rises with weight	8
sigma (Σ)	add up; it comes with a counter and the counter's first and last values	1
size (of a matrix)	rows by columns: a matrix with 3 rows and 2 columns is 3 × 2, said "3 by 2"	11
stable, unstable (equilibrium)	stable if a state pushed a little away from it returns to it; unstable if the state moves further away	8
subscript	a small letter or number set low after a symbol; part of the symbol's name, and never a multiplication	1
sum and difference rules	the derivative of a sum or difference of two functions is the sum or difference of their derivatives	3
time constant (τ)	ρ divided by the total slope, in days; the gap to the equilibrium shrinks as e^(-t/τ), so τ is 1 divided by the rate constant k	8
vector	an ordered list of numbers, called its entries; one row of a data table is a vector	10
weighted sum	each vector multiplied by its own number, then added: cv + dw for vectors v and w and numbers c and d	10
weights (of a weighted sum)	the numbers a weighted sum multiplies by; "weight" here means a multiplier, not body weight	10
where-clause	the words that fix what each symbol in an equation stands for, often a clause beginning "where" after it; it fixes each symbol's meaning, its unit and its sign	1

The series: all 196 books

In build order: Book 0 first, then every Introductory rung, then every Intermediate rung, and so on; within a level, a subject comes after the subjects it depends on. "done" means that book's version 1 is finished. This is Book 2.

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OBESITY EXPERTISE · BOOK 2

Mathematics for modelling and inference · Rung 1

A short book of mathematics for reading research on obesity. It covers the rate of change and the running total worked out rather than named, the number e, a table of numbers and what multiplying by one does, a random quantity with its average and spread, and the point where a changing quantity settles. Then it shows how to read an equation symbol by symbol, and how to follow a derivation line by line without jumping to the result.

Book 2 of the series' 196 books, free to read at drharshmaheshwari.com.



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